# Funding Liquidity Risk From a Regulatory Perspective

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## Funding Liquidity Risk from a Regulatory Perspective

### Abstract

In the Basel regulation the required capital of a financial institution is based on conditional measures of the risk of its future equity value such as Value-at-Risk, or Expected Shortfall. In Basel 2 the uncertainty on this equity value is captured by means of changes in asset prices (market risk) and default of borrowers (credit risk), and mainly concerns the asset component of the balance-sheet. Our paper extends this analysis by taking also into account the funding and market liquidity risks. The latter risks are consequences of changes in customers or investors' behaviors and usually concern the liability component of the balance sheet. In this respect our analysis is in the spirit of the most recent Basel 3 and Solvency 2 regulations. Our analysis highlights the role of the different types of risks in the total required capital. Our analysis leads to clearly distinguish defaults due to liquidity shortage and defaults due to a lack of solvency and, in a regulatory perspective, to introduce two reserve accounts, one for liquidity risk, another one for solvency risk. We explain how to fix the associated required capitals.

**Keywords:** Regulation, Funding Liquidity Risk, Liquidity Shortage, Solvency 2, Value-at-Risk, Asset/Liability Management.

# 1 Introduction

In Basel 2 regulation the computation of the required capital is based on conditional measures of risk of a future portfolio value, such as the Value-at-Risk (VaR), or the Expected Shortfall (ES). This future value is defined on a crystallized portfolio: the portfolio allocation it kept fixed, whereas the prices are uncertain. This practice accounts for the uncertainty on market prices (i.e. the market risk) as well as for the counterparty risk (i.e. the default risk). However, the assumption of crystallized portfolio has to be discussed more carefully: first, a financial institution modifies its portfolio allocation according to the expected price movements; second, this allocation can be subject to changes in the behaviors of customers and investors. The first reason for varying allocation depends on the portfolio management by the institution. It is endogenous and mainly concerns the asset component of the balance sheet. The second source of varying allocation is exogenous and concerns the liability component of the balance sheet. In this paper we focus on these exogenous shocks on portfolio allocation and we analyze how they can be taken into account in risk measures, required capital and asset-liability management.

We implicitly consider the need for cash, that is the funding liquidity risk, when we account for these exogenous shocks on portfolio allocation. Thus, from a regulatory perspective, we are going from a Basel 2 approach to a Basel 3/Solvency 2 approach common to banks and insurance companies.

Our analysis highlights funding and market liquidity risks as new factors for predicting the default of financial institutions. Moreover, we show that it is important to disentangle the liquidity and solvency sources of default. This led to the introduction of two default boundaries, one for liquidity and one for default. This extension has to be compared with the standard literature on relation between rollover risk and default risk for instance, which assumes that a unique default boundary is given [see e.g. Black, Cox (1976), Leland (1994), Leland, Toft (1996), He, Xiang (2012)]. Such an assumption is compatible with the basic analysis of solvency risk [Merton (1974)], but does not capture the specificity of liquidity risks. We also carefully study the link between liquidity risk and default risk in the historical world. This allows to describe the different possible joint regimes of liquidity and solvency for a given firm. Loosely speaking, each regime is associated with a joint rating of the firm. The first rating is for liquidity risk, the second one for solvency risk. Finally we consider the regulatory perspective. We explain that two reserve accounts have to be introduced, one for hedging default due to liquidity shortage and a second one for hedging default due to a lack of solvency. We discuss how to jointly manage these accounts, and last, but not least, how to fix the associated required capitals.

The plan of the paper is as follows. In Section 2, we consider simplified balance sheets, how they are impacted by the exogenous shocks on quantities and how they will be quickly adjusted to avoid a short term default due to a funding liquidity shortage. We first analyze the case of a financial institution with an unlimited line of credit. In this framework, a lack of cash after the shock requires the use of the credit line and an additional cost. When the credit line is of limited size, the financial institution can be obliged to sell in a hurry illiquid assets. This has an additional cost, but also possibly an effect on these prices, if the exogenous shocks are in the same direction for a number of financial institutions. In other words, the funding liquidity risk can generate a market liquidity risk and this effect be reinforced by the liquidity spiral highlighted in [Brunnermeier, Pedersen (2009)].

Sections 3 and 4 focus on the risk measures. Whereas under Basel 2 the risk measures were accounting for the risk on prices (market and default risks) only, in our framework we have also to account for the risk on exogenous shocks in allocation and the co-risk between prices and exogenous shocks in allocation. We first consider how the standard Value-at-Risk (VaR) can be decomposed to highlight the effect of the funding and market liquidity risks. We also introduce specific measures of the probability of using the credit line and of the magnitude of this use as well as measures of the probability of selling illiquid assets and the size of these sells. Section 5 discusses the definition and the design of reserves. This new framework requires two reserves accounts in order to control for both the solvency risk and the liquidity risk. We derive the levels of required capital associated with these accounts. In other words, we provide the bidimensional Value-at-Risk appropriate to our problem. Section 6 concludes. Proofs are gathered in appendices.

# 2 The balance sheets and their responses to exogenous shocks on liability

In this section, we consider a simplified balance sheet, which can be impacted by exogenous shocks on the allocation of its liability component. These shocks can be due to the behavior of depositors or of holders of saving accounts for banks, to the lapse and new demand of life insurance contracts for insurance companies, to the demand and sell by investors for funds... We first describe the balance sheet at date $t$ before the exogenous shock, then the effect of the shock on this balance sheet. Finally, we discuss how the institution can quickly react according to the existing constraints on its credit line.

## 2.1 The balance sheet

Let us consider an institution with the simplified balance sheet at date $t$ represented in Table 1.

| Asset | Liability |
|---|---|
| $x_{1,t}p_{1,t}$ | $L_{1,t}$ |
| | $L_{0,t}$ |
| $x_{0,t}$ | 0 |
| | $Y_t$ |

Table 1: Balance sheet at date $t$ with positive cash balance

The asset component of the balance sheet includes:

- a quantity $x_{1,t}$ of an illiquid asset with unitary value $p_{1,t}$. This value can be a market price if this asset is traded on the market, or an actuarial value if this is a bond for instance. Because of the lack of liquidity, trading a large quantity of this asset quickly can have an effect on its unitary value.

- a value of cash $x_{0,t}$. Cash is a riskfree asset with riskfree rate assumed equal to zero.

The liability component of the balance sheet includes:

- $L_{1,t}$, which is the value of the long term part of the debt. We assume that the elements of this debt are far from time-to-maturity and that the debtholders of that debt cannot ask for prepayment. Finally we assume a debt with payment in fine to avoid to account for intermediate coupons.

- $L_{0,t}$ is the part of the debt sensitive to funding liquidity risk. It includes long term debt close to maturity, but also debt whose time-to-maturity is uncertain and whose repayment can be asked for at any time such as deposits.

- 0 is the amount of debt on a credit line with interest rate $\gamma$. This is a standard credit line, which can be used used without collateral.

- $Y_t$ is the equity, that is the shareholders' (accounting) value.

In the balance sheet of Table 1, we assume that the bank has a positive quantity of cash and no debt on its credit line. What occurs if the amount of debt in the credit line is strictly positive? Since there is no return on cash and the debt on credit line is costly, there is an arbitrage opportunity and the bank has interest in getting the debt on its credit line as small as possible by (partly) reimbursing it in cash. Thus only the balance between cash and debt on credit line matters. If this cash balance $x_{0,t}$ is positive, we get the scheme in Table 1. Otherwise, the balance sheet is given in Table 2.

| Asset | Liability |
|---|---|
| $x_{1,t}p_{1,t}$ | $L_{1,t}$ |
| | $L_{0,t}$ |
| 0 | $-x_{0,t}$ |
| | $Y_t$ |

Table 2: Balance sheet at date $t$ with negative cash balance

Such a balance sheet can be summarized in different ways. For instance:

- the total asset at date $t$ is: $A_t = x_{1,t}p_{1,t} + max(x_{0,t}, 0)$,
- the equity value at date $t$ is:

$$\begin{aligned} Y_t &= A_t - L_{1,t} - L_{0,t} - max(-x_{0,t}, 0) \\ &= x_{1,t}p_{1,t} - L_{1,t} - L_{0,t} + x_{0,t}, \end{aligned}$$

- the leverage or debt/equity ratio is: $\tau_t = \dfrac{L_{1,t} + L_{0,t} + max(-x_{0,t}, 0)}{Y_t}$.

## 2.2 The exogenous shocks and their consequences on the balance sheet (unlimited credit line)

The balance sheet is exposed to exogenous shocks on both prices and quantities. The exogenous shocks on the asset side only concern the unitary value of the illiquid asset:

$$p_{1,t} \longrightarrow p_{1,t+1}. \tag{1}$$

Since the price is positive, this shock is restricted by:

$$\delta p_{1,t+1} \equiv p_{1,t+1} - p_{1,t} \geq -p_{1,t}. \tag{2}$$

The exogenous shocks on the liability side concern both the long and short term debts:

$$L_{1,t} \longrightarrow L_{1,t+1} \text{ and } L_{0,t} \longrightarrow L_{0,t+1}. \tag{3}$$

These shocks on debts are subject to different restrictions:

$$\delta L_{1,t+1} \equiv L_{1,t+1} - L_{1,t} \geq 0, \tag{4}$$

$$\delta L_{0,t+1} \equiv L_{0,t+1} - L_{0,t} \in [-L_{0,t}, \infty). \tag{5}$$

The long term debt can only increase, whereas the short term debt can either increase, or decrease. They imply a new demand or supply for cash. In particular, $\delta L_{t+1} = \delta L_{0,t+1} + \delta L_{1,t+1} > -L_{0,t}$, and $L_{0,t}$ is the (maximal) exposure of the institution to funding liquidity risk.

In our simplified framework the shocks on debt admit different interpretations.
i) They can be due to the rollover risk on the debt. When a debt matures, the bank is not necessarily successful in issuing a new debt with the same face value and possibly a different term-to-maturity. There is a debt capacity. This debt capacity may depend on the asset held by the financial institution, which might be seen as collateral [see Schleifer, Vischny (1992), or Acharya et al. (2011)]. When the short term debt is large and the debt capacity becomes small, the rollover risk can become huge [see Morris, Shin (2010) and Appendix 1 for a comparison]. For instance, in the mid-March 2008, Bear Stearns had an exposure to rollover risk of about $ 85 billions on the overnight market [Cohan (2009)]. Similarly in the months before its bankruptcy, Lehman-Brothers was rolling 25% of its debt every day through overnight repos [see the discussion in Brunnermeier (2009), Krishnamurthi (2010)].
ii) A part of the long term debt becomes close to maturity, and is transferred from $L_1$ to $L_0$. This effects is neglected in our approach where $L_{1,t+1} \geq L_{1,t}$.
iii) We may have demands for an early reimbursement of the debt, depositor runs, or lapses of life insurance contracts (for insurance companies). This a redemption risk.

To summarize the shocks on debt capture a mix of rollover risk and redemption risk without identifying their relative magnitudes. The exogeneity assumption concerning redemption risk is standard in the literature [see e.g. the exogenous Poisson liquidity shocks assumed in He, Xiang (2012), or the so-called EBIT model in Goldstein et al. (2009), Hackarth et al. (2006)]. The exogeneity assumption for the rollover component corresponds to the assumption of rather illiquid debt market and of exogenous debt capacity. Due to this rationing on debt issuing ,the liquidity needs due to these exogenous shocks on debt will be fulfilled by using sequentially the available cash, the credit line and then some asset sales.
Other assumptions have also to be commented with respect to the literature.
i) There is no exogenous shock on the equity. Typically, if the institution were a hedge fund, we assume equal to zero the withdrawal of investors. This other type of redemption risk is disregarded. Symmetrically, we assume that the institution in distress cannot quickly raise new equity.
ii) The interest rate on the credit line is constant and in particular independent of the risk of default of the institution. Thus we disregard the so-called "margin call risk".

Let us first assume that the institution has an unlimited credit line. After these shocks on debts, the institution can have enough cash to offset the potential short term debt withdrawals. Otherwise, this institution will withdraw on its credit line with a rate $\gamma > 0$, keeping crystallized the quantity invested in illiquid assets. Just after these shocks, the cash balance becomes:

$$\tilde{x}_{0,t+1|t} \equiv x_{0,t} + \delta L_{1,t+1} + \delta L_{0,t+1}, \tag{6}$$

where the index notation $t+1|t$ conditional on $t$ is to mention that the allocation $x_{1,t}$ is crystallized. This cash balance does not account for the cost of the credit line. The cost adjusted cash balance is:

$$x_{0,t+1|t} = \begin{cases} \tilde{x}_{0,t+1|t}, & \text{if } \tilde{x}_{0,t+1|t} \geq 0, \\ (1+\gamma)\tilde{x}_{0,t+1|t}, & \text{if } \tilde{x}_{0,t+1|t} < 0, \end{cases} \tag{7}$$

assuming that the interest on the credit line is immediately paid. The new balance sheets are given in Tables 3 or 4 depending on the sign of the cash balance.

| Asset | Liability |
|---|---|
| $x_{1,t}p_{1,t+1}$ | $L_{1,t+1}$ |
| | $L_{0,t+1}$ |
| $x_{0,t} + \delta L_{1,t+1} + \delta L_{0,t+1}$ | $0$ |
| | $Y^*_{t+1\|t}$ |

Table 3: Balance sheet at date $t+1$ with positive cash balance

| Asset | Liability |
|---|---|
| $x_{1,t}p_{1,t+1}$ | $L_{1,t+1}$ |
| | $L_{0,t+1}$ |
| | $-(1+\gamma)(x_{0,t} + \delta L_{1,t+1} + \delta L_{0,t+1})$ |
| | $Y^*_{t+1\|t}$ |

Table 4: Balance sheet at date $t+1$ with negative cash balance

We deduce the expression of the equity value:

$$Y^*_{t+1|t} = x_{1,t}p_{1,t+1} + x_{0,t} - L_{1,t} - L_{0,t} - \gamma\left(-x_{0,t} - \delta L_{1,t+1} - \delta L_{0,t+1}\right)^+, \tag{8}$$

$$= Y_t + x_{1,t}\delta p_{1,t+1} - \gamma\left(-x_{0,t} - \delta L_{1,t+1} - \delta L_{0,t+1}\right)^+. \tag{9}$$

Note that $Y^*_{t+1|t}$ is generally different from $Y_{t+1}$, since the institution has not yet adjusted its portfolio allocation (especially $x_{1,t}$) to take into account the change in prices, that is the market risk. If the interest rate $\gamma$ on the credit line is equal to zero, the changes in the liability components are neutral for the institution and the change in equity value is just the change in the value of the risky part of the portfolio, that is, $x_{1,t}(p_{1,t+1} - p_{1,t})$. If $\gamma$ is strictly positive, there is a cost for the lack of cash proportional to the need for cash. As noted above, this future equity value is for a portfolio which is partially crystallized, since the quantity of illiquid asset has been kept fixed.

## 2.3 The exogenous shocks and their consequences on the balance sheet (limited credit line)

Let us now extend the analysis by considering a limited credit line: the bank can not borrow more than a given amount $M$, say, at rate $\gamma$. In order to include this new feature, we need to enable the bank to sell its illiquid asset. This fire sale is done with an haircut denoted by $H$ (with $0 < H < 1$) assumed independent of the size of the sale, and of the price. Thus the value of a fire sale differs from its accounting value.
As previously, the bank is not short of cash when:

$$x_{0,t} + \delta L_{1,t+1} + \delta L_{0,t+1} \geq 0. \tag{10}$$

When the bank is short of cash, two regimes can arise. They depend on the position of the liquidity shortage $-x_{0,t} - \delta L_{1,t+1} - \delta L_{0,t+1}$ with respect to the limit $\tilde{M} = M/(1+\gamma)$ of the credit line including the immediate payment of interest. If $-x_{0,t} - \delta L_{1,t+1} - \delta L_{0,t+1} < \tilde{M}$, the credit line is large enough to satisfy both the need for cash and the payment of the interest on the credit line; the new balance sheet is similar to the balance sheet corresponding to an unlimited credit line. If the liquidity shortage is higher than $\tilde{M}$, the bank covers the cash balance by selling illiquid asset. The quantity of illiquid asset needed is:

$$\frac{-x_{0,t} - \delta L_{1,t+1} - \delta L_{0,t+1} - \tilde{M}}{H p_{1,t+1}}. \tag{11}$$

The new balance sheet is given in Table 5.

| Asset | Liability |
|---|---|
| $x_{1,t}p_{1,t+1} - \frac{1}{H}\left(-x_{0,t} - \delta L_{1,t+1} - \delta L_{0,t+1} - \tilde{M}\right)$ | $L_{1,t+1}$ |
| | $L_{0,t+1}$ |
| 0 | $(1+\gamma)\tilde{M}$ |
| | $\tilde{Y}^{*}_{t+1\vert t}$ |

Table 5: Balance sheet at date $t+1$ with activation of the credit line and sell of the illiquid asset

Therefore, the equity value becomes:

$$\begin{aligned}\tilde{Y}^{*}_{t+1|t} = Y_t + x_{1,t}\delta p_{1,t+1} - \gamma min\left[\tilde{M}; (-x_{0,t} - \delta L_{1,t+1} - \delta L_{0,t+1})^{+}\right] \\ - \left(\frac{1}{H} - 1\right)\left(-x_{0,t} - \delta L_{1,t+1} - \delta L_{0,t+1} - \tilde{M}\right)^{+}.\end{aligned} \tag{12}$$

To meet its liquidity shortage, why does the bank use first the credit line and then sell the illiquid asset? Actually, we have implicitly assumed that the cost of the haircut on selling illiquid asset, $\frac{1}{H} - 1$, is larger than the interest rate of the credit line $\gamma$, so that selling illiquid asset first and then using the credit line is sub-optimal.[3]

[3]This simplified sequence to meet the liquidity shortage does not take into account the possibility to get liquidity via the Central Bank by using illiquid asset as collateral. In fact we simply assume that such a possibility will be opened after the use of the reserve accounts introduced for liquidity risk.

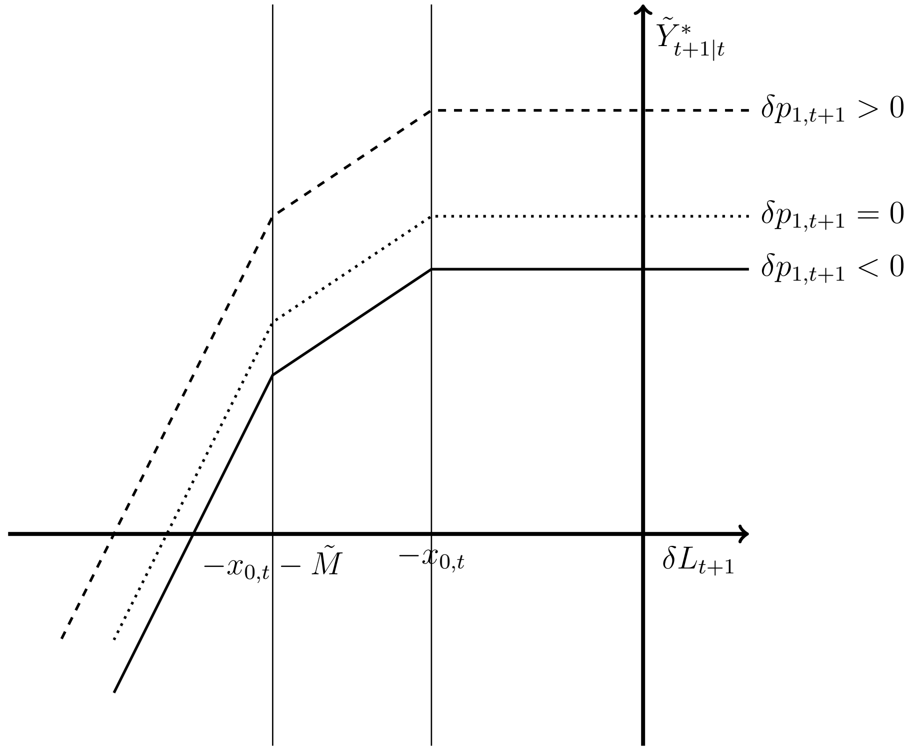


Figure 1: Equity Value with Tranches on the Liquidity Shortage (with $x_{0,t} > 0$)
Solid (respectively, dotted and dashed) line is the equity value as a function of the liquidity shock for a negative (respectively, null and positive) shock on price.

By comparing the expressions of $\tilde{Y}^*_{t+1|t}$ with limited credit line [see eq. (12)] and unlimited credit line [see eq. (9)], we see that the call on liquidity shortage with strike 0 is now replaced by a combination of tranches written on the liquidity shortage, taking into account the different marginal costs of liquidity (see Figure 1).

For small liquidity shortage ($\tilde{M} = M = \infty$), the two expressions are equal. For large liquidity shortage with a limited credit line, the cost is higher since the selling of illiquid asset (with an haircut) may heavily penalize the equity value.

It may even occur that the liquidity shortage is so large that the bank has not enough illiquid asset to sell to reimburse its funding needs. This regime arises when:

$$\left(-x_{0,t} - \delta L_{1,t+1} - \delta L_{0,t+1} - \tilde{M}\right)^+ > x_{1,t}(p_{1,t} + \delta p_{1,t+1})H. \tag{13}$$

In this situation, the bank sells all its illiquid asset and then defaults: the debts are not repaid in full and the equity value is zero, $Y^*_{t+1|t} = 0$, under the assumption of limited liability of the shareholders.

Combining the different cases, the equity value of the bank is:

$$Y^*_{t+1|t} = \tilde{Y}^*_{t+1|t} \mathbb{1}_{\left(-x_{0,t} - \delta L_{1,t+1} - \delta L_{0,t+1} - \tilde{M}\right)^+ < x_{1,t}(p_{1,t} + \delta p_{1,t+1})H}. \tag{14}$$

# 3 Profit and Loss (P&L) distribution (unlimited credit line)

From eq. (9), the P&L is given by:

$$Y^*_{t+1|t} = Y_t + x_{1,t}\delta p_{1,t+1} - \gamma\left(-x_{0,t} - \delta L_{1,t+1} - \delta L_{0,t+1}\right)^+ . \tag{15}$$

If $\gamma = 0$, we get the standard P&L computed as if the liability component of the balance sheet and the allocations of the asset component were crystallized. In this case, the price change $\delta p_{1,t+1}$ is the only variable which is unknown at date $t$ and generates the uncertainty of the P&L.

If $\gamma > 0$, an additional component captures the cost of cash. This component is uncertain, since both $L_{1,t+1}$ and $L_{0,t+1}$ are stochastic too.

The P&L distribution can be summarized in different ways. In particular, it is possible to disentangle the asset (price) and liability (quantities) components of the risk, or to focus on the need for cash.

## 3.1 Decomposition of the Value-at-Risk

Let us first consider the VaR associated with the P&L. This $VaR_t(\alpha, \gamma)$ is such that:

$$\mathbb{P}_t\left[Y^*_{t+1|t} < -VaR^\alpha_t(\gamma)\right] = \alpha, \tag{16}$$

where $\mathbb{P}_t$ denotes the probability conditional on the information available at date $t$ and $\alpha$ is the risk level, such as 1% or 5%. This risk measure accounts for the uncertainty on both price $p_1$ and quantities $L_1$ and $L_0$. When $\gamma = 0$, $VaR^\alpha_t(0)$ is such that:

$$\mathbb{P}_t\left[Y_t + x_{1,t}\delta p_{1,t+1} < -VaR^\alpha_t(0)\right] = \alpha. \tag{17}$$

$VaR_t(\alpha, 0)$ is the standard VaR at risk level $\alpha$ under Basel 2, which accounts for price change only. Moreover, since the P&L depends negatively on the rate $\gamma$ on the credit line, by construction we get:

$$VaR^\alpha_t(\gamma) > VaR^\alpha_t(0), \quad \forall\alpha. \tag{18}$$

Thus the short term VaR, $VaR^\alpha_t(\gamma)$ can be decomposed as:

$$VaR^\alpha_t(\gamma) = VaR^\alpha_t(0) + \left[VaR^\alpha_t(\gamma) - VaR^\alpha_t(0)\right], \tag{19}$$

where $VaR^\alpha_t(0)$ is the standard VaR and the non-negative difference $VaR^\alpha_t(\gamma) - VaR^\alpha_t(0)$ provides the additional cost in terms of VaR of the exogenous shocks on the liability component of the balance sheet. Note that even if $VaR^\alpha_t(\gamma)$ is an interesting measure of risk on the P&L, we will see in Section 5 that it does not provide enough information to define the required capitals needed for hedging the liquidity and solvency risks, respectively.

## 3.2 The need for the credit line

By analogy with the regulation of default risk, we can summarize the need for cash by considering the probability of using the credit line (this is the analogue of the probability of default) and the expected use of this credit line when it is used (this is the analogue of the expected loss given default). The probability of use is:

$$PU_t = \mathbb{P}_t \left[ x_{0,t} + \delta L_{1,t+1} + \delta L_{0,t+1} < 0 \right], \tag{20}$$

and the expected use given use is:

$$EUGU_t(\gamma) = (1+\gamma)\mathbb{E}_t \left[ -x_{0,t} - \delta L_{1,t+1} - \delta L_{0,t+1} | - x_{0,t} - \delta L_{1,t+1} - \delta L_{0,t+1} > 0 \right]. \tag{21}$$

## 3.3 Illustration

Let us illustrate how these summary statistics of the risks, that are the VaR, the PU and the EUGU, depend on the initial situation of the bank and on the distribution of the shocks on price and quantity. To facilitate the comparison with the standard literature and to get closed form expressions, we assume Gaussian shocks. With such a Gaussian specification, we do not take into account the different inequality restrictions existing for the price and the short term and long term debts (see the constraints (2), (4) and (5) on the exposures to funding liquidity risk). Then only the total change in debt matters. Thus the Gaussian assumption is:

$$\begin{pmatrix} \delta p_{1,t+1} \\ \delta L_{1,t+1} + \delta L_{0,t+1} \end{pmatrix} \sim \mathcal{N}\left( \begin{pmatrix} \mu_p \\ \mu_L \end{pmatrix} ; \begin{pmatrix} \sigma_p^2 & \rho\sigma_p\sigma_L \\ \rho\sigma_p\sigma_L & \sigma_L^2 \end{pmatrix} \right). \tag{22}$$

The initial situation of the bank at date $t$ is characterized by the allocation $x_{0,t}$, $x_{1,t}$ and by the equity value $Y_t$.

### 3.3.1 Value-at-Risk

Let us first consider the Value-at-Risk. It is proved in Appendix 2, that $VaR_t^\alpha(\gamma)$ is the solution of:

$$\alpha = \mathbb{E}_t \left\{ \Phi \left( \frac{\gamma(-x_{0,t} - \mu_L - \sigma_L U)^+ - VaR_t^\alpha(\gamma) - Y_t - x_{1,t}\left(\mu_p + \rho\sigma_P U\right)}{x_{1,t}\sigma_p\sqrt{1-\rho^2}} \right) \right\}, \tag{23}$$

where $U$ is a standard normal variable independent of the information available at date $t$.

When $\gamma = 0$, this equation simplifies to the standard formula of a Gaussian VaR:

$$\alpha = \Phi\left( \frac{1}{\sigma_p} \left( \frac{-VaR_t^\alpha(0) - Y_t}{x_{1,t}} - \mu_p \right) \right), \tag{24}$$

which is equivalent to:

$$VaR_t^\alpha(0) = -Y_t - x_{1,t}\left(\mu_p + \sigma_p \Phi^{-1}(\alpha)\right). \tag{25}$$

Otherwise, the value $VaR_t^\alpha(\gamma)$ has to be computed numerically.

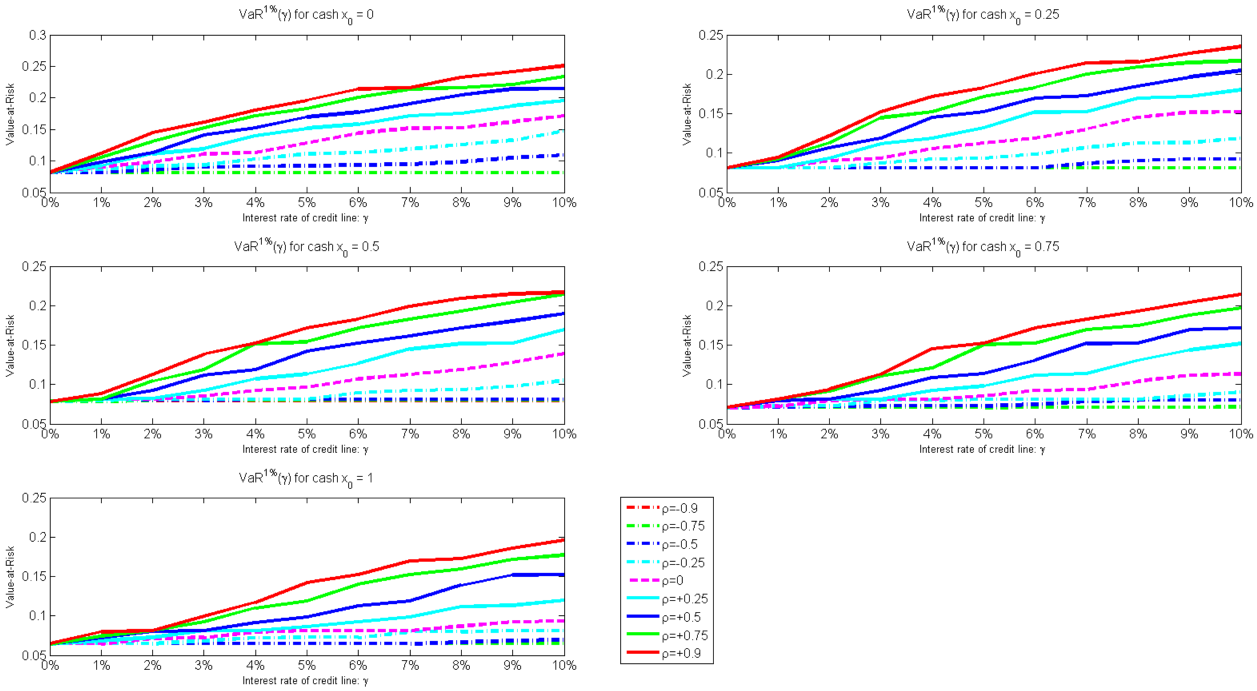


Figure 2: Values of the VaR for different values of $\gamma$, $\rho$ and different levels of cash.

As an illustration, let us consider the following situation:

The general size of the balance sheet is given by $L_{1,t} + L_{0,t} = 100$ and $Y_t = 5$. The volume of liquid asset, $x_{0,t}$, varies from 0 to 1 (with a step of 0.25). We normalize the unitary value of illiquid asset by setting $p_{1,t} = 1$, so that the volume of illiquid asset $x_{1,t}$ is deduced based on the previous balance sheet components. For simplicity, we consider a dynamics of shocks without drift by setting $\mu_L = \mu_p = 0$. The correlation $\rho$ ranges between $-0.9$ to $+0.9$. The last parameters are the variance of shocks : $\sigma_L$ and $\sigma_p$. We set $\sigma_L$ to get a probability of use of the credit line $PU$ (see below) equal to 0.1%. We set $\sigma_p = \sigma_L/(L_{1,t} + L_{0,t}) = \sigma_L/100$ to get similar risks on the asset and liability sides.

Figure 2 displays the sensitivity of the VaR at 1% for various situations. The 5 panels provides the evolution of the VaR with respect to the interest rate $\gamma$ for a given volume of cash and a given correlation $\rho$. The VaR seems rather linear in the interest rate $\gamma$, or even constant. The slope of this quasi linear function can be approximated by considering the Taylor expansion of the VaR in a neighborhood of $\gamma = 0$, or equivalently by computing the sensitivity of the VaR to the credit line interest rate $\dfrac{\partial VaR_t^\alpha(0)}{\partial \gamma}$. We have (see Appendix 3):

$$\frac{\partial VaR_t^\alpha(0)}{\partial \gamma} = \mathbb{E}\left[(-x_{0,t} - \delta L_{1,t+1} - \delta L_{0,t+1})^+ | - Y_t - x_{1,t}\delta p_{1,t+1} = VaR_t^\alpha(0)\right]. \qquad (26)$$

The values of this sensitivity are provided in Table 6.

We provide in Figure 3 the evolution of the VaR with respect to the correlation between asset and liability shocks with different values of $\gamma$. Each panel displays the results for

| Corr. Cash | $x_0 = 0$ | $x_0 = 0.25$ | $x_0 = 0.5$ | $x_0 = 0.75$ | $x_0 = 1$ |
|---|---|---|---|---|---|
| $\rho = -0.90$ | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| $\rho = -0.75$ | 0.2 | 0.3 | 0.8 | 0.6 | 0.0 |
| $\rho = -0.50$ | 25.5 | 10.1 | 2.5 | 7.9 | 4.5 |
| $\rho = -0.25$ | 60.2 | 34.2 | 23.9 | 18.0 | 15.0 |
| $\rho = \ 0$ | 81.9 | 65.0 | 55.0 | 38.4 | 26.3 |
| $\rho = +0.25$ | 103.9 | 90.3 | 82.6 | 73.7 | 50.1 |
| $\rho = +0.50$ | 121.1 | 112.2 | 100.9 | 91.6 | 79.9 |
| $\rho = +0.75$ | 139.1 | 123.6 | 123.1 | 114.8 | 102.5 |
| $\rho = +0.90$ | 154.0 | 139.8 | 125.7 | 130.3 | 119.9 |

Table 6: The Sensitivity of $VaR^{1\%}(0)$ with respect to $\gamma$.

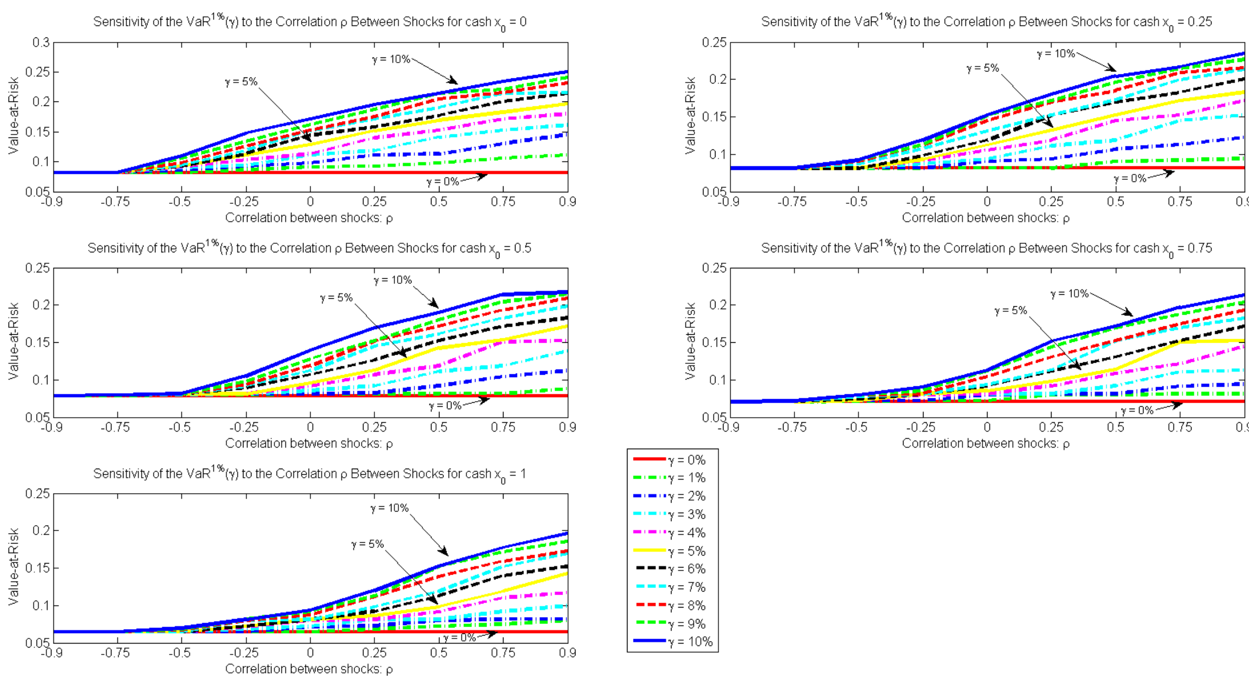


Figure 3: Evolution of the $VaR^{1\%}(\gamma)$ with respect to the Correlation Between Asset and Liability Shocks.

a given level of cash. We observe a non-linear response to the correlation between the shocks as well as a spreading of the VaR for various interest rate when the correlation is high. When the shocks are anti-correlated, the interest rate does not matter so much since there is a diversification between the asset and liability risks.

### 3.3.2 Need for cash

Let us finally consider the need for cash. In a Gaussian framework (see Appendix 2), the probability of use $PU_t$ and the expected use given use $EUGU_t$ of the credit line are:

$$PU_t = \Phi\left(\frac{-x_{0,t} - \mu_L}{\sigma_L}\right), \tag{27}$$

and

$$EUGU_t(\gamma) = -(1+\gamma)\left(x_{0,t} + \mu_L - \sigma_L \frac{\varphi\left(\frac{-x_{0,t} - \mu_L}{\sigma_L}\right)}{\Phi\left(\frac{-x_{0,t} - \mu_L}{\sigma_L}\right)}\right), \tag{28}$$

where $\Phi$ (resp., $\varphi$) is the cumulative density function (resp., probability density function) of a standard Gaussian distribution and the Mill's ratio $\varphi/\Phi$ captures the selectivity bias [see Heckmann, (1979)]. Whereas the total VaR [see eq. (24)] depends on the joint distribution of the risks on price and quantities, the $PU$ and $EUGU$ summary statistics are specific of the funding liquidity risk and depend on the distribution of $\delta L$ only.

# 4 P&L distribution (limited credit line)

## 4.1 Description of the regimes

From equations (12)-(14), the P&L is given by:

$$\begin{aligned} Y^*_{t+1|t} = \Big( & Y_t + x_{1,t}\delta p_{1,t+1} - \gamma min\left[\tilde{M}; (-x_{0,t} - \delta L_{1,t+1} - \delta L_{0,t+1})^+\right] \\ & - \left(\frac{1}{H} - 1\right)(-x_{0,t} - \delta L_{1,t+1} - \delta L_{0,t+1} - \tilde{M})^+\Big) \mathbb{1}_{\left(-x_{0,t} - \delta L_{1,t+1} - \delta L_{0,t+1} - \tilde{M}\right)^+ < x_{1,t} p_{1,t+1} H}. \end{aligned} \tag{29}$$

Different regimes may occur depending on the magnitude of asset and liability shocks. We index the regimes with ratings measuring the degree of liquidity and solvency distress of the financial institution. First, we identify four regimes for liquidity, according to the differences between the need for cash and the resources of cash:

- $\mathcal{R}^\ell(AA)$: if $-x_{0,t} < \delta L_{1,t+1} + \delta L_{0,t+1}$, the bank has enough cash to absorb the liquidity shock. There is no need to activate the credit line or to sell illiquid assets.

- $\mathcal{R}^\ell(A)$: if $-x_{0,t} - \tilde{M} < \delta L_{1,t+1} + \delta L_{0,t+1} < -x_{0,t}$, the bank activates the credit line. The cash provided by the credit line is sufficient to meet the liquidity need of the bank.

- $\mathcal{R}^\ell(B)$: if $-x_{0,t} - \tilde{M} - x_{1,t}H(p_{1,t} + \delta p_{1,t+1}) < \delta L_{1,t+1} + \delta L_{0,t+1} < -x_{0,t} - \tilde{M}$, the credit line is fully activated and there is enough illiquid asset to be sold.
- $\mathcal{R}^\ell(D)$: if $\delta L_{1,t+1} + \delta L_{0,t+1} < -x_{0,t} - \tilde{M} - x_{1,t}H(p_{1,t} + \delta p_{1,t+1})$, even with a full activation of the credit line and selling all the illiquid asset, the bank cannot gather enough cash to meet its liquidity requirement. In this situation, the bank will default for liquidity reasons.

Second, two solvency regimes can be defined depending on the sign of the equity.

- $\mathcal{R}^\mathcal{S}(A)$: if $\Big(Y_t + x_{1,t}\delta p_{1,t+1} - \gamma min\left[\tilde{M}; (-x_{0,t} - \delta L_{1,t+1} - \delta L_{0,t+1})^+\right] - \left(\frac{1}{H} - 1\right)(-x_{0,t} - \delta L_{1,t+1} - \delta L_{0,t+1} - \tilde{M})^+\Big) > 0$, the bank is solvent.
- $\mathcal{R}^\mathcal{S}(D)$: if $\Big(Y_t + x_1\delta p_{1,t+1} - \gamma min\left[\tilde{M}; (-x_{0,t} - \delta L_{1,t+1} - \delta L_{0,t+1})^+\right] - \left(\frac{1}{H} - 1\right)(-x_{0,t} - \delta L_{1,t+1} - \delta L_{0,t+1} - \tilde{M})^+\Big) < 0$, the bank is insolvent.

These solvency regimes are defined ex-post, that is once the demand for liquidity has been satisfied. These regimes differ from the ex-ante solvency regimes usually considered in Basel 2 regulation. These ex-ante solvency regimes would depend on the sign of the ex-ante equity: $Y^{\star\star}_{t+1} = x_{1,t}p_{1,t+1} + x_{0,t} - L_{1,t+1} - L_{0,t+1}$.

The liquidity and solvency dimensions are crossed to define up to 7 regimes. We denote $\mathcal{R}(\text{liquidity rating}, \text{solvency rating})$ the regime where $\mathcal{R}^\ell(\text{liquidity rating})$ and $\mathcal{R}^\mathcal{S}(\text{solvency rating})$ overlap. Let us describe these regimes:

- Regime $\mathcal{R}(AA, A)$: the bank is solvent and has enough cash to face the liquidity shock. This regime corresponds to a business-as-usual situation.
- Regime $\mathcal{R}(A, A)$: the bank is solvent, but has to activate a part of its credit line to meet its liquidity need. This situation is typical of a negative shock on the asset liability side combined with a normal (or positive) situation on the asset side.
- Regime $\mathcal{R}(B, A)$: the bank is solvent, but its liquidity need is so important that the credit line is not sufficient. To cover its need for cash, the bank is forced to sell a fraction of its illiquid assets with an haircut. This situation arises when the liability shock is huge while there is a significant gain-in-value on the illiquid asset. In fact, to remain solvent even with selling its illiquid asset, it is necessary that the illiquid asset has a sufficiently high price.
- Regime $\mathcal{R}(AA, D)$: the bank is insolvent without any cash difficulties. This regime reflects a severe loss on illiquid asset. The default is only due to a pure solvency matter: the liquidity aspect is not involved.
- Regime $\mathcal{R}(A, D)$: the bank is insolvent and meets its liquidity need by activating its credit line. The default is at first glance due to solvency imbalance. However, it may happen that the imbalance derives from the additional cost generated by the activation of the credit line. This cost is the interest of the credit line $\gamma$ which captures the cost of time.

- Regime $\mathcal{R}(D, D)$: the bank is insolvent and unable to meet its liquid need even with activating its credit line and selling all its illiquid assets. In this situation, the solvency reason and the liquidity reason are intertwined to account for the default. It is typically the case when the bank suffers from both a large liquidity shock (e.g. massive deposit withdraw) and a large asset shock (e.g. massive drop in price of illiquid asset).

- Regime $\mathcal{R}(B, D)$: the bank is insolvent, but meets its liquidity needs. As for Regime $\mathcal{R}(A, D)$, the liquidity may have played a role (through the additional cost induced by the activation of the credit line), but the default is mainly due to solvency matter. This situation may be characteristic of a bank that sell so much of its illiquid asset that it becomes insolvent. The withdrawing creditors have been fully repaid while the staying creditors suffer a loss: it is the "first-come, first-served" principle.

The Regime $\mathcal{R}(D, A)$ is impossible. Indeed if the bank is in default for a liquidity reason, its ex-post equity is negative and it is automatically also insolvent.

This result is compatible with the observation that firms can be in default due to a liquidity shortage, whereas still in good health when the default risk is measured by the standard distance-to-default implemented by the Moody's KMV corporation [see e.g. Davydenko (2007)]. Indeed the distance-to-default is based on the asset value $x_{1,t}p_{1,t+1}$ evaluated at the market price prior to liquidity shocks and for the crystallized allocation, which does not account for the potential sale of a fraction of the portfolio to finance the need for cash (see Section 5.1). This asset value, often called fundamental value of the firm in the literature [Leland, Toft (1996), Brunnermeier, Pedersen (2009), He, Xiong (2012)], is prior to liquidity shocks. In our framework the financial institution has first to fulfill the need for cash, and the solvency is treated after this liquidity step. It is important to distinguish these two notions of solvency, before and after the treatment of this "senior debt" to be immediately reimbursed.

**Proposition 1** (Number of regimes). *The seven regimes arise if and only if* $Y_t < x_{1,t}p_{1,t} + \gamma\tilde{M}$ *and* $L_{0,t} > x_0 + \tilde{M}$.

*Proof.* See Appendix 5. □

The two conditions are written on the initial structure of the balance-sheet. The first one corresponds to the usual balance-sheet of a financial institution: the equity is a small fraction of the liability while most assets are risky. The second condition means that the financial institution cannot face a complete withdrawal of its short-term funding with its cash and its credit line[4]. Figure 4 represents the seven regimes with the shock on total debt on the $x-$axis and the shock on price on the $y-$axis. The dashed lines represent the frontiers of liquidity regimes (activation of the credit line, selling of the illiquid assets, depletion of illiquid assets to sell) while the solid red line represents the solvency frontier: above this frontier, the bank is solvent and the bank is insolvent, otherwise. Dotted lines are extensions of lines out of the feasible set.

The slope of the solvency line when the credit line is activated (in the center of Figure 4) is proportional to $\gamma$: this slope captures the cost of time. When the bank sells illiquid

[4] The other cases are discussed in Appendix 4.

asset (on the left part of Figure 4), the slope of the solvency line is higher: the increase is proportional to $\frac{1-H}{H} - \gamma$, and captures the cost of market illiquidity.

## 4.2 Decomposition of the Value-at-Risk

Let us consider the VaR associated with the P&L. Comparing to the case with unlimited credit line, the VaR now depends on three parameters, which are the two costs of liquidity shortage, $\gamma$ and $1/H$, and the size $M$ of the credit line. This $VaR_t^\alpha(\gamma, M, H)$ is such that:

$$\mathbb{P}_t\left[Y^*_{t+1|t} < -VaR_t^\alpha(\gamma, M, H)\right] = \alpha, \tag{30}$$

where $\mathbb{P}_t$ denotes the probability conditional on the information available at date $t$. This risk measure accounts for the uncertainty on both price $p_1$ and quantities $L_1$ and $L_0$.

When $M = \infty$, the VaR has been decomposed into two terms: the standard VaR (without liquidity feature) and a term accounting for liquidity risk. When $M < \infty$, the liquidity risk generates two terms for the use of the credit line and for the sell of illiquid asset, respectively. The $VaR_t^\alpha(\gamma, M, H)$ is decomposed as:

$$\begin{aligned} VaR_t^\alpha(\gamma, M, H) = {} & VaR_t^\alpha(0, \infty, H) \\ & + \left[VaR_t^\alpha(\gamma, \infty, H) - VaR_t^\alpha(0, \infty, H)\right] \\ & + \left[VaR_t^\alpha(\gamma, M, H) - VaR_t^\alpha(\gamma, \infty, H)\right]. \end{aligned} \tag{31}$$

The first term, $VaR_t^\alpha(0, \infty, H)$ is the standard VaR under Basel 2, which accounts for price change only. The second term: $VaR_t^\alpha(\gamma, \infty, H) - VaR_t^\alpha(0, \infty, H)$, is non-negative and represents the additional cost associated with an unlimited credit line. The third term: $VaR_t^\alpha(\gamma, M, H) - VaR_t^\alpha(\gamma, \infty, H)$, represents the additional cost when selling illiquid asset due to the limitation of the credit line. This term is non-negative as far as interest rate $\gamma$ is lower than the opportunity cost $\frac{1-H}{H}$.

This decomposition (31) of the VaR interpreted as the global cost of default is the analogue in the physical world of the common practice of decomposing the firm's credit spread into a "liquidity premium" and a "default premium" components [see e.g. Longstaff *et al.* (2005), Beber et al. (2009), Schwarz (2009)]. Such an additive decomposition of a credit risk premium has been criticized, since it cannot capture the price of interaction between liquidity and solvency risks. A similar criticism applied to decomposition (31). The mistake is to use a global VaR to measure a bivariate risk. We explain in Section 5 how to solve this problem.

## 4.3 The need for the credit line

As previously, we can define the probability of use of the credit line $PU_t$ and the expected use given use $EUGU_t$:

$$PU_t = \mathbb{P}_t\left[0 < -x_{0,t} - \delta L_{1,t+1} - \delta L_{0,t+1}\right], \tag{32}$$

and the expected use given use is:

$$EUGU_t = (1+\gamma)\mathbb{E}_t\left[-x_{0,t} - \delta L_{1,t+1} - \delta L_{0,t+1} | 0 < -x_{0,t} - \delta L_{1,t+1} - \delta L_{0,t+1} < \tilde{M}\right]. \tag{33}$$

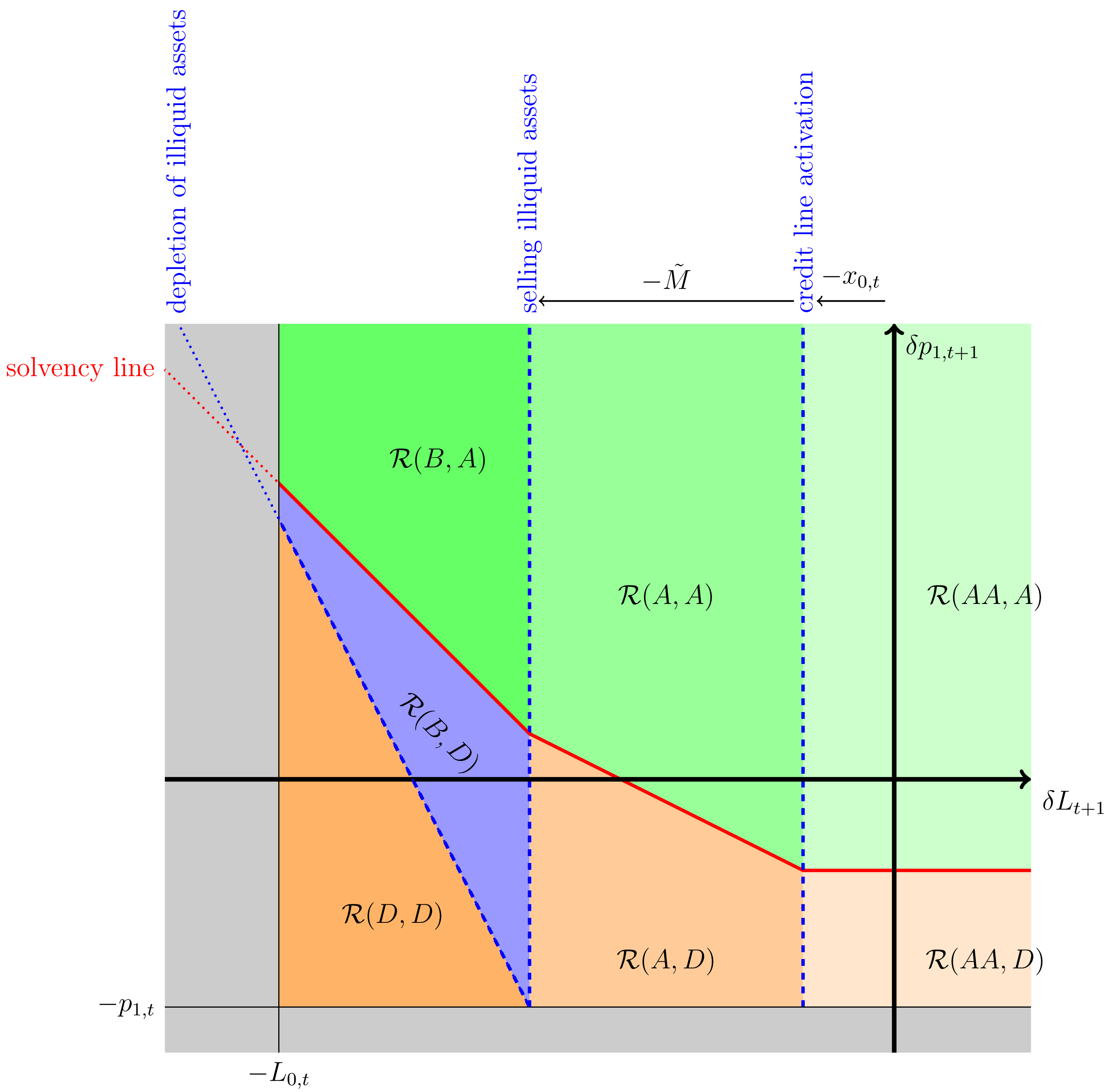


Figure 4: 7 Regimes with Limited Credit Line

Since the credit line is limited, we have:

$$EUGU_t \leq (1+\gamma)\tilde{M} = M. \tag{34}$$

### 4.4 The need to sell illiquid asset

We can also introduce the probability to sell illiquid assets (or equivalently the probability that the credit line is not sufficiently large) $PS_t$ and the expected volume of sell of illiquid asset to be sold $ESGS_t$.

The probability to sell illiquid assets is:

$$PS_t = \mathbb{P}\left[\tilde{M} < -x_{0,t} - \delta L_{1,t+1} - \delta L_{0,t+1}\right], \tag{35}$$

and the expected volume of illiquid asset to be sold is:

$$ESGS_t = \frac{1-H}{H}\mathbb{E}\left[-x_{0,t} - \delta L_{1,t+1} - \delta L_{0,t+1} - \tilde{M} | \tilde{M} < -x_{0,t} - \delta L_{1,t+1} - \delta L_{0,t+1} < \tilde{M} + x_{1,t}p_{1,t+1}H\right] \tag{36}$$

The limitation on the volume of illiquid asset to sell leads us to define the probability that the bank goes bankrupt for liquidity shortage $PB_t$. We have:

$$PB_t = \mathbb{P}\left[\tilde{M} + x_{1,t}p_{1,t+1}H < -x_{0,t} - \delta L_{1,t+1} - \delta L_{0,t+1}\right]. \tag{37}$$

The events defining the probabilities of use $PU_t$, of sell $PS_t$ and of bankruptcy $PB_t$ are nested and then we get:

$$PB_t < PS_t < PU_t. \tag{38}$$

For instance, the expected use of the credit line is: $EU_t = (PU_t - PS_t) \times EUGU_t$.

These summary statistics are detailed in the Gaussian case in Appendix 4.

## 5 The reserves for solvency and funding liquidity risk

The different regimes following exogenous shocks on debt and price have been described in Section 4 without any regulation. In this section we discuss how reserves can be introduced to partly control the risks of default due to either a lack of liquidity, or a solvency problem. We first recall the standard definition of reserve in Basel 2. Then we extend this framework. Indeed two reserve accounts have now to be introduced to manage the two types of default. We discuss how they can be jointly used by the regulator and we explain how to compute the required capital.

### 5.1 The standard definition of reserve

In the standard analysis of solvency risk, the portfolio is crystallized and the debt is assumed to be predetermined. The balance sheets including the reserve account (on the asset side) are given in Table 7 before and after the shocks on price:

| Asset | Liability |
|---|---|
| $x_{1,t}p_{1,t}$ | $L_{1,t}$ |
| | $L_{0,t}$ |
| $x_{0,t}$ | |
| $R_t$ | |
| | $Y_t$ |

before the change in price

| Asset | Liability |
|---|---|
| $x_{1,t}p_{1,t+1}$ | $L_{1,t}$ |
| | $L_{0,t}$ |
| $x_{0,t}$ | |
| $R_t$ | |
| | $Y_{t+1}$ |

after the change in price

Table 7: Balance sheet with a reserve account

Then in the standard regulation, the level of reserve $R_t$ is fixed to get a small probability of failure. When this probability is fixed to $\alpha$, the level of reserve is the solution of:

$$\mathbb{P}_t\left[Y_{t+1} < 0\right] = \alpha \tag{39}$$

$$\iff \quad \mathbb{P}_t\left[x_{1,t}p_{1,t+1} + x_{0,t} + R_t - L_{1,t} - L_{0,t} < 0\right] = \alpha \tag{40}$$

$$\iff \quad \mathbb{P}_t\left[x_{1,t}\delta p_{1,t+1} + (x_{1,t}p_{1,t} + x_{0,t} - L_{1,t} - L_{0,t}) < -R_t\right] = \alpha \tag{41}$$

The level of reserve $R_t$ is set to the (lower) $\alpha-$quantile of the conditional distribution of $x_{1,t}\delta p_{1,t+1} + Y_t^*$, with $Y_t^* = x_{1,t}p_{1,t} + x_{0,t} - L_{1,t} - L_{0,t}$ being the difference between the equity at date $t$ and the reserve level. This computation shows that the reserve for solvency risk is directly linked to a VaR.

In the presentation above, the reserves are represented on the asset side. It is possible to consider that the reserves are a part of equity as the label "required capital" suggests. The total equity, $Y_t$, is the sum of the required capital $R_t$ and an excess of capital $\overline{Y}_t$: $Y_t = R_t + \overline{Y}_t$. The regulation imposes that the excess of capital is positive: $\overline{Y}_t > 0$. However, this excess of capital has a counterpart on the asset side which must not impact the riskiness of the financial institution (otherwise, the required capital would increased). It can be written as a total of cash $\overline{x}_{0,t} = x_{0,t} + R_t$, with the restriction that the financial institution has access only to a fraction of this cash, $x_{0,t}$ while the other part $R_t$ is locked. In this alternative perspective, the balance sheet is given in Table 8.

| Asset | Liability |
|---|---|
| $x_{1,t}p_{1,t}$ | $L_{1,t}$ |
| | $L_{0,t}$ |
| $\overline{x}_{0,t}$ | |
| | $R_t$ |
| | $\overline{Y}_t$ |

Table 8: Balance sheet with required capital

## 5.2 The extended framework

Let us now extend the standard capital requirement by introducing two types of reserves to hedge the two types of defaults for liquidity shortage and lack of solvency, respectively.

| Asset | Liability |
|---|---|
| $x_{1,t}p_{1,t}$ | $L_{1,t}$ |
| | $L_{0,t}$ |
| $x_{0,t}$ | 0 |
| $R_{1,t}$ | |
| $R_{2,t}$ | |
| | $Y_t$ |

Table 9: The Initial Balance Sheet with Two Reserve Accounts

### 5.2.1 The initial balance sheet

The situation is very different when we want to account also for the funding liquidity risk. Indeed, as seen in Section 4, the institution can default because of a liquidity shortage as well as because of a lack of solvency. These two causes of default require two types of reserve to be treated in an appropriate way. The balance sheet before the shocks on prices and quantities is described in Table 9.

There are two reserve accounts $R_{1,t}$ and $R_{2,t}$, say. Reserve account $R_{1,t}$ if for the treatment of liquidity risk, whereas reverse account $R_{2,t}$ augmented for the residual of $R_{1,t}$ is for the treatment for solvency risk. First, these reserve accounts correspond to required capitals. Therefore, the initial equity has to be larger than the total required capital : $Y_t > R_{1,t}+R_{2,t}$. Equivalently, the excess of capital at date $t$ is positive: $\overline{Y}_t \equiv Y_t - R_{1,t} - R_{2,t} > 0$. Second, these reserve accounts are composed of perfectly liquid asset (ie. riskfree in our model). However these reserve accounts differs from the cash $x_{0,t}$. The account $x_{0,t}$ can be used freely by the financial institution for responding to the funding liquidity shocks. The reserve accounts can be used by the institution only with the authorization of the supervisor. Consequently, for the financial institution, regardless any intervention of the supervisor, the reserve accounts have a perfect market liquidity, but have a perfect funding illiquidity.

For expository purpose, we have kept the same notations as in Table 1. But of course, the institution has to adjust its portfolio in order to satisfy the introduction of the two reserve accounts. The type of adjustment is clearly out of the scope of this paper[5].

### 5.2.2 Supervisor's intervention

We consider that the financial institution has first to satisfy its liquidity need, and then its solvency situation is assessed. For sake of simplicity, we consider that the supervisor acts in last resort: the financial institution get access to the reserve $R_{1,t}$ only when it has already fully used its credit line and sold all its illiquid assets. If a fraction of the reserve $R_{1,t}$ is sufficient, only this fraction is unlocked.

When it comes to solvency, three situations can be identified. First, if the equity is negative, the financial institution is insolvent. Second, when both the equity and the excess of capital are positive, the financial institution is solvent and fulfills the regulatory

[5]However, for the determination of the levels of reserves, it is assumed that the introduction of the reserve account is financed by an identical increase of capital (see Section 5).

constraint. Third, it may happen that the bank has positive equity, but no positive excess of capital. In this latter situation, the financial institution is in distress. Note that the intervention concerning liquidity does not necessarily interfer with the solvency step.

Let us review the regimes described in Section 4 when there are two reserve accounts.

**i) Liquidity regimes**

We can now identify five liquidity regimes. Whereas regimes $\mathcal{R}^\ell(AA)$, $\mathcal{R}^\ell(A)$, $\mathcal{R}^\ell(B)$ are unchanged, in case of a liquidity problem the supervisor can unlock the cash reserve. A new regime $\mathcal{R}^\ell(C)$, say, arises when the cash reserve is needed and sufficient:

$$-x_{0,t}-\tilde{M}-R_{1,t}-x_{1,t}H(p_{1,t}+\delta p_{1,t+1}) < -\delta L_{1,t+1}-\delta L_{0,t+1} < -x_{0,t}-\tilde{M}-x_{1,t}H(p_{1,t}+\delta p_{1,t+1}). \tag{42}$$

The remaining amount of reserves after the intervention of the supervisor is denoted by $R^*_{1,t}$, and defined by:

$$R^*_{1,t} = min\left[R_{1,t}; \left(-x_{0,t} - \tilde{M} - x_{1,t}H(p_{1,t} + \delta p_{1,t+1}) - \delta L_{1,t+1} - \delta L_{0,t+1}\right)^+\right]. \tag{43}$$

When the cash reserve is not sufficient, the default regime $\mathcal{R}^\ell(D)$ arises:

$$-\delta L_{1,t+1} - \delta L_{0,t+1} < -x_{0,t} - \tilde{M} - R_{1,t} - x_{1,t}H(p_{1,t} + \delta p_{1,t+1}). \tag{44}$$

**ii) Solvency regimes**

Similarly, we get an additional solvency regime corresponding to a situation with positive equity below the regulatory constraint. Therefore we have three solvency regimes:

- regime $\mathcal{R}^\mathcal{S}(A)$ arises when the financial institution's equity is above the regulatory constraint.
- regime $\mathcal{R}^\mathcal{S}(C)$ arises when the financial institution's equity is positive, but below the regulatory constraint.
- regime $\mathcal{R}^\mathcal{S}(D)$ arises when the financial institution's equity is negative.

**iii) Combining regimes of liquidity and regimes of solvency**

As in the analysis presented in Section 4, the five liquidity regimes are crossed with the three solvency regimes to define up to 15 regimes. The standard situation involves less than 15 regimes:

**Proposition 2.** *The situation with 11 regimes represented in Figure 5 arises if and only if $Y_t < x_{1,t}p_{1,t} + \gamma\tilde{M}$ and $L_{0,t} > x_0 + \tilde{M}$.*

*Proof.* See Appendix 6. □

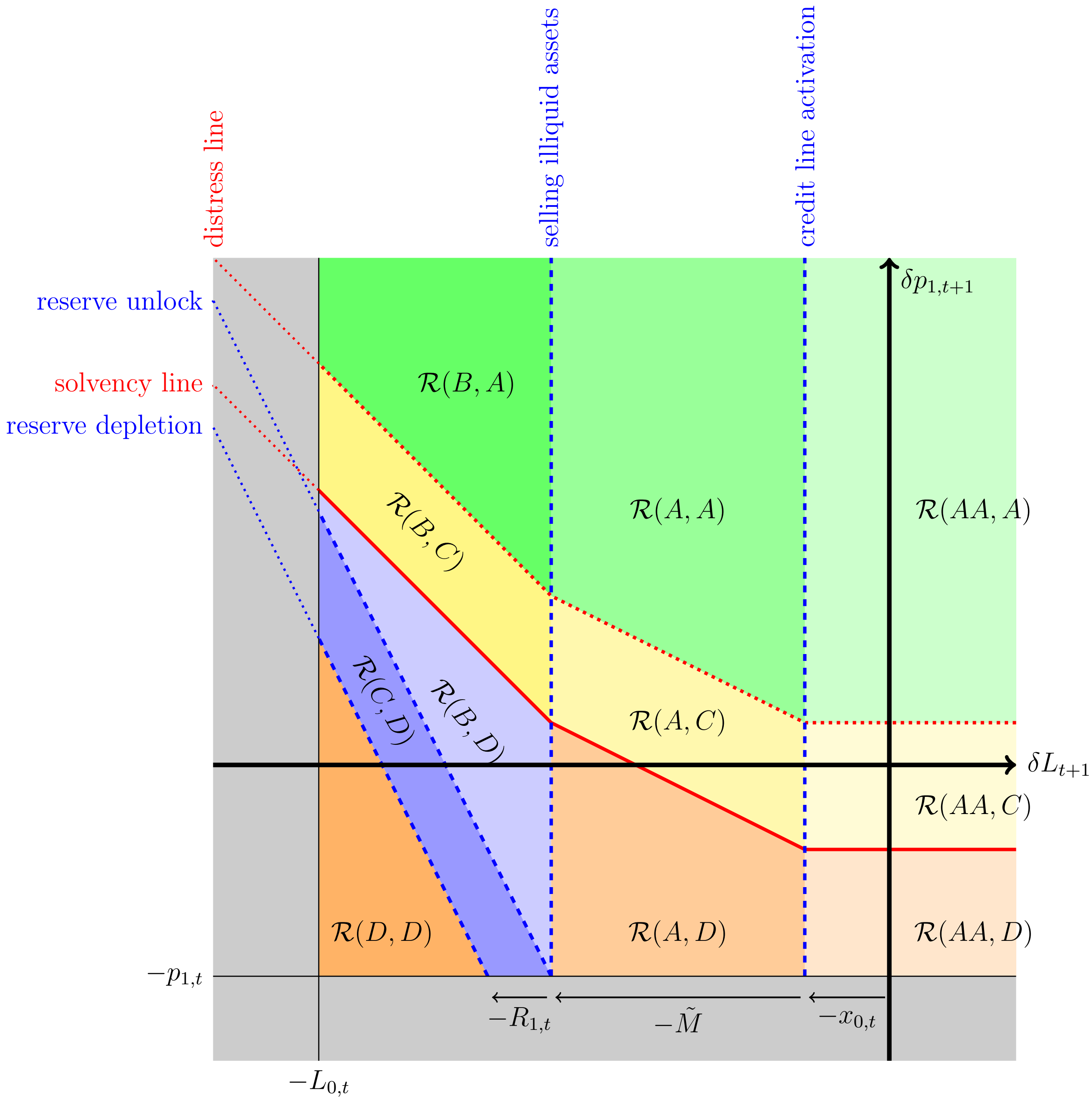


Figure 5: 11 Regimes with Limited Credit Line and Reserves

### 5.2.3 Impact of reserves

Figure 5 represents the 11 regimes combining solvency and liquidity. For instance, Regime $\mathcal{R}(A, C)$ corresponds to a financial institution with positive equity, but lower than the required one, that activated its credit line to solve its liquidity shortage. Regime $\mathcal{R}(C, D)$ corresponds to the supervisor's intervention: the reserve $R_{1,t}$ is unlocked to face the massive liability shock experienced by the institution.

There are two modifications when comparing Figure 5 (with reserves) and Figure 4 (without reserves). The first modification is the "distress line" (dashed red line) which introduces a second level of equity. This line does not cross regimes when there are liquidity difficulties: it is only a solvency point of reference. The second difference is the downward extension of the blue sector (or equivalently, a downward withdraw of Regime $\mathcal{R}(D, D)$). This extension is due to the supervisor's intervention. The intervention increases the capacity of the financial institution with respect to liquidity balance. This modification is only a matter of liquidity; it does not affect the solvency status of the financial institution.

In our framework, it is possible to define typical situations that describe the health of a financial institution and accounts for risk classes. We propose to consider four main regions in our analysis (see Figure 6):

- A financial institution is *alive* when it fulfills the regulatory constraint [Regimes $\mathcal{R}(AA, A)$, $\mathcal{R}(A, A)$ and $\mathcal{R}(B, A)$]. This region is characteristic of business-as-usual.

- A financial institution is *in distress* when it is solvent, but does not fulfill the regulatory constraint [Regimes $\mathcal{R}(AA, C)$, $\mathcal{R}(A, C)$ and $\mathcal{R}(B, C)$]. This region is like a buffer along the previous region: it is not as good as alive, but is close to it.

- A financial institution is *in default* when it is insolvent but does not need to sell its illiquid asset to face its liquidity need [Regimes $\mathcal{R}(AA, D)$ and $\mathcal{R}(A, D)$]. With this definition, default refers to solvency difficulties only. In this region, a massive fall in price makes the equity buffer insufficient.

- A financial institution is *in resolution* when selling illiquid asset leads to its insolvency [Regimes $\mathcal{R}(B, D)$ and $\mathcal{R}(C, D)$]. This domain is the symmetric of the previous one: the difficulties on the liability side are beyond the buffer capacity.

- A financial institution is *bankrupt* when it has not enough illiquid asset to be sold [Regime $\mathcal{R}(D, D)$]. This region is characteristic of very adverse shocks on the asset and liability sides. Here, it is difficult to distinguish a major reason between liquidity and solvency.

Other partitions of regimes may be considered, and be more adapted to other specific concerns. Moreover, our partition depends on the intervention rules. We have adopted a role of "lender of last resort" to avoid as much as possible to introduce moral hazard features. As a consequent, the supervisor intervenes only when the financial is insolvent and only focuses on liquidity matters. If a supervisor is concerned only the well functioning of short-term market (*i.e.* minimize interbank market dry-up.), this intervention rule seems suitable. However, one could definitely argue that supervisor may step in earlier in

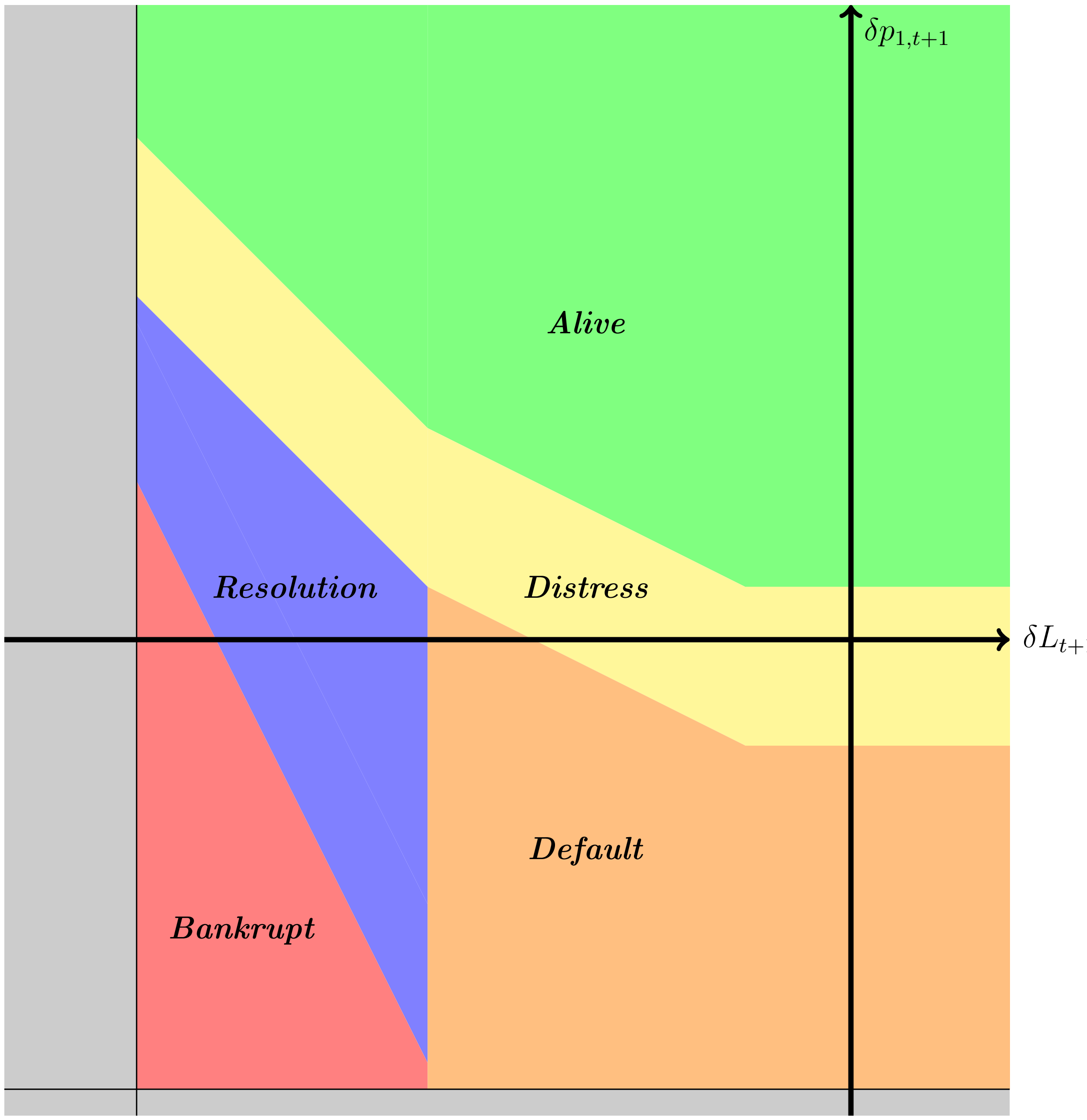


Figure 6: Majors situations to describe an institution

order to avoid the insolvency, or at least to cushion the impact on the real economy. The objective of supervision intervention could be to avoid the fire selling of illiquid assets that decrease economy financing and may trigger fire sell spiral for other institutions. In this perspective, the supervisors would unlock the reserve as soon as the cost of selling illiquid asset is "unreasonable". Defining what is "unreasonable" goes clearly beyond our paper, but lay stem from current debates on banking resolution and more generally on systemic risk.

## 5.3 Computation of reserves

How to fix the two reserve levels in practice? Intuitively, by trying to control the two probabilities of default, that are:

$$\mathbb{P}_t\,[\text{default due to funding liquidity}] \equiv PD_t^F(R_{1,t}, R_{2,t}), \tag{45}$$

$$\text{and } \mathbb{P}_t\,[\text{default due to lack of solvency}] \equiv PD_t^S(R_{1,t}, R_{2,t}). \tag{46}$$

With Figure 5, it is possible to define explicitly the events underlying these two probabilities:

- "default due to funding liquidity" is covered by $\mathcal{R}(D, D)$,
- "default due to lack of solvency" is covered by the union of $\mathcal{R}(B, D)$, $\mathcal{R}(C, D)$, $\mathcal{R}(A, D)$, $\mathcal{R}(AA, D)$.

For given levels of the probabilities of default, $\alpha_1$, $\alpha_2$, say, we will have to solve the bivariate system of equations:

$$\begin{cases} PD_t^F(R_{1,t}, R_{2,t}) &= \alpha_1, \\ PD_t^S(R_{1,t}, R_{2,t}) &= \alpha_2, \end{cases} \tag{47}$$

whose solutions $R_{1,t}$, $R_{2,t}$ will depend on $\alpha_1$, $\alpha_2$, on the initial structure of balance sheet and on the joint distribution of the changes in price and quantities. In particular both $R_{1,t}$ and $R_t = R_{1,t} + R_{2,t}$ will lose the standard interpretation in terms of Value-at-Risk.

## 5.4 Computation of reserves: illustration

As previously, we consider a Gaussian framework where the distribution of the exogenous shocks is given by:

$$\begin{pmatrix} \delta p_{1,t+1} \\ \delta L_{1,t+1} + \delta L_{0,t+1} \end{pmatrix} \sim \mathcal{N}\left( \begin{pmatrix} \mu_p \\ \mu_L \end{pmatrix}; \begin{pmatrix} \sigma_p^2 & \rho\sigma_p\sigma_L \\ \rho\sigma_p\sigma_L & \sigma_L^2 \end{pmatrix} \right). \tag{48}$$

We apply these shocks to a balance sheet very sensitive to liquidity shocks with a small amount of cash, a small credit line, a large haircut... Such extreme situations have been encountered before the recent crisis. For instance Almeida et al. (2012) indicate that firms with large amount of debt maturing after the third quarter of 2007 cut investment

| | $R_2 = 0$ | $R_2 = 1$ | $R_2 = 2$ | $R_2 = 3$ | $R_2 = 4$ | $R_2 = 5$ |
|---|---|---|---|---|---|---|
| $R_1 = 0$ | 2.59% | 2.59% | 2.59% | 2.59% | 2.59% | 2.59% |
| $R_1 = 1$ | 2.05% | 2.05% | 2.05% | 2.05% | 2.05% | 2.05% |
| $R_1 = 2$ | 1.60% | 1.60% | 1.60% | 1.60% | 1.60% | 1.60% |
| $R_1 = 3$ | 1.23% | 1.23% | 1.23% | 1.23% | 1.23% | 1.23% |
| $R_1 = 4$ | 0.94% | 0.94% | 0.94% | 0.94% | 0.94% | 0.94% |
| $R_1 = 5$ | 0.72% | 0.72% | 0.72% | 0.72% | 0.72% | 0.72% |

Table 10: Probability of default due to funding liquidity $PD_t^F$

| | $R_2 = 0$ | $R_2 = 1$ | $R_2 = 2$ | $R_2 = 3$ | $R_2 = 4$ | $R_2 = 5$ |
|---|---|---|---|---|---|---|
| $R_1 = 0$ | 38.36% | 35.36% | 32.48% | 29.8% | 27.31% | 24.97% |
| $R_1 = 1$ | 35.89% | 33.02% | 30.34% | 27.85% | 25.51% | 23.40% |
| $R_1 = 2$ | 33.48% | 30.8% | 28.3% | 25.96% | 23.86% | 21.91% |
| $R_1 = 3$ | 31.16% | 28.67% | 26.33% | 24.23% | 22.27% | 20.53% |
| $R_1 = 4$ | 28.96% | 26.62% | 24.51% | 22.56% | 20.81% | 19.26% |
| $R_1 = 5$ | 26.84% | 24.74% | 22.79% | 21.04% | 19.49% | 18.12% |

Table 11: Probability of default due to lack of solvency $PD_t^S$

to capital ratio by one third of their pre-crisis levels. Similarly the heavy use of short-term debt financing is a key factor in the collapse of Bear Stearns and Lehman-Brothers [Brunnermeier (2009), Krishnamurthy (2010)].

The general size of the balance sheet is given by $L_{1,t}+L_{0,t} = 100$ and $x_t = x_{1,t}p_{1,t} = 102$ (with a normalized unitary value of illiquid asset $p_{1,t} = 1$). The cash $x_{0,t}$ is 5. The credit line is designed with $M = 1$ and $\gamma = 10\%$. We consider an important haircut on illiquid assets since $H = 10\%$. Since $R_{1,t}$ and $R_{2,t}$ will be varying, the initial equity value is defined as $Y_t = x_{1,t}p_{1,t} + x_{0,t} + R_{1,t} + R_{2,t} - L_{1,t} - L_{0,t}$. For simplicity, we consider a dynamics of shocks without drift by setting $\mu_L = \mu_p = 0$, and consider a correlation between shocks: $\rho = 0.5$. The shock on liability has a magnitude $\sigma_L = 10$. In order to have similar risk on the asset and liability side, we impose $\sigma_p = \sigma_L/(L_{1,t} + L_{0,t})$.

Table 10 (respectively Table 11) reports the evolution of the probability of default due to funding $PD_t^F$ (resp. due to a lack of solvency $PD_t^S$). The probability of default due a lack of solvency $PD_t^S$ is decreasing in both reserve accounts $R_1$ and $R_2$. The reserve account $R_1$ which can be unlocked in case of liquidity difficulty has an impact on the probability of default due to funding, whereas this probability of default is insensitive to the level of the reserve account $R_2$. Then these tables have to be inverted to derive the levels of reserve $R_1$ and $R_1$ as function of the risk levels $\alpha_1$ and $\alpha_2$.

# 6 Conclusion

Basel new regulation puts forward liquidity risk, as solvency risk was previously tackled. In this paper, we have developed a stylized balance-sheet that accounts for market risk and liquidity risks with both market liquidity risk and funding liquidity risk. The first risk is captured by a stochastic price of (illiquid) assets, the second risk by an hair-cut on illiquid asset when turns into cash (liquidation cost), and the third risk by stochastic volume on the liability side. The focus is on the liquidity management since the banks can deal its liquidity shortage by using a cash reserve, activating a credit line, or selling its illiquid assets. The presence of several distinct sources of risk led us to extend the risk indicators and the required levels of reserve that usually only accounts for market risk. This extension is twofold. First we introduce a decomposition of the VaR into a market risk VaR, a market liquidity risk premium and a funding liquidity risk premium. Second, we propose to introduce two types of reserve for a more accurate management of the default due to liquidity shortage and the default for lack of solvency, respectively. The two reserve levels might be followed overtime for the financial institutions and insurance companies under regulation in order to monitor jointly the two types of risks.

Our framework relies on simple assumptions: the credit line has a unique cost or the hair-cut is independent of sell volume. Moreover, the supervisor's intervention correspond to a last-resort behavior. However, it is easy to extend our model by introducing more sophisticated features. The interest rate paid for the credit line may depend on the called volume. The hair-cut on illiquid asset may become an increasing function of the sell volume to include a price-impact component.

More sophisticated rules for the intervention of the supervisor may also be introduced. Typically, our approach has not distinguished the systematic and unsystematic components of the risks. Four reserve accounts will have to be introduced to also manage the systematic liquidity risk and the systematic solvency risk, which are not necessarily driven by a single systematic factor. The analysis with four reserve accounts[6] would be much more complicated since it requires the joint analysis of several balance sheets including the contagion schemes [see e.g. Cifuentes, Ferruci, Shin (2005) or Gai, Haldane, Kapadia (2011)]. Moreover, the rule for unlocking the reserve for systematic and unsystematic liquidity risks will now depend on the existence and magnitude of fire sales of illiquid assets by several institutions simultaneously. Such an extension is left for future research.

Another extension would be the introduction of a central bank providing liquidity against collateral, especially in a situation of high systemic liquidity risk. The aim of such a central bank intervention would be to breakdown the liquidity spiral. This extension, including the optimal level of intervention of the central bank is also out of scope of our paper.

Finally there exists a theoretical literature to analyze the optimal capital structure of a firm, to see how this structure, the optimal leverage and the default threshold are endogenously fixed, and to understand the conflicts between managers and equity holders [see e.g. Leland (1994), He, Xiong (2012), Bolton et al. (2011)]. It would be interesting

[6]Monitoring four reserve accounts for a financial institution is not a complex task since a genuine balance sheet has hundreds of lines. In our framework, most of these lines are gathered under the illiquid asset class.

to see how this literature would be modified in presence of reserve accounts and with a regulator as the third actor.

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# 1 Appendix: Comparison with Morris and Shin (2010)

Our model shares several features with Morris and Shin (2010). However, we distance ourselves from this paper in various respects.
The balance sheet proposed in Morris and Shin (2010) is similar to ours (see Table 12) with two classes of assets (risky/illiquid and non-risky cash) and two classes of debts (short and long term). We both consider that short term debt holders may ask for repayment. This liquidity needs can be covered by cash and by using the risky/illiquid asset.

| Assets | Liabilities |
|---|---|
| Cash $M$ | Equity $E_2$ |
| Risky Asset $\theta_2 Y$ | Short Debt $S_2$ |
| | Long Debt $L_2$ |

Table 12: Balance sheet of the bank at the final date in Morris and Shin (2010)

In our model, the risky assets are assumed to be sold, whereas Morris and Shin propose to use them as collateral to raise new debt in the baseline model : "[their] assumption is in contrast to the usual assumption in models of bank runs where the bank has to liquidate long-term assets ("dig up potatoes planted in the field") to pay the early withdrawers" (p6). Contrasting with Morris and Shin, we authorize the financial institution to activate a credit line as an intermediate solution after using its own cash and before selling its illiquid asset.
As highlighted by Morris and Shin, when the hair-cut on the illiquid assets depends on the realized value of the illiquid asset (as in the extension 6.2 of their baseline model), the insolvency risk and the illiquidity risk get intertwined. Since in our model the hair cut is applied to the current value of the illiquid asset, we include this feature.

The key difference between our model and Morris and Shin's one corresponds to the hinging between shocks on the asset and liability sides. In their framework, there is first a shock on the asset side. Then short-term debt holders decide to roll-over or not on the short-term debt. This choice builds on comparing the expected return of rolling over and a reserve return with game theory features. In other word, the shock on the short-term debt volume is a deterministic, albeit sophisticated, function of the shocks on the asset side.
On the contrary, we do not impose a deterministic relationship between the shocks on the asset and liability sides. For instance, in the illustration, we use a simple bivariate Gaussian distribution with a correlation parameter. However, this joint distribution can be specified to tend to Morris and Shin's framework as limiting case.

# 2 Appendix: Computation for an unlimited credit line

Let us first provide some useful results concerning the truncated Gaussian distribution (see e.g. ):

**Lemma 1.** *Let us consider a Gaussian variable $X \sim \mathcal{N}(\mu, \sigma^2)$:*

$$i) \qquad \mathbb{P}\left[X < \alpha\right] = \Phi\left(\frac{\alpha - \mu}{\sigma}\right) \tag{a.1}$$

$$ii) \qquad \mathbb{E}\left[X | X < \alpha\right] = \mu - \sigma \frac{\varphi\left(\frac{\alpha - \mu}{\sigma}\right)}{\Phi\left(\frac{\alpha - \mu}{\sigma}\right)} \tag{a.2}$$

$$iii) \qquad \mathbb{E}\left[X | \beta < X < \alpha\right] = \mu + \sigma \frac{\varphi\left(\frac{\beta - \mu}{\sigma}\right) - \varphi\left(\frac{\alpha - \mu}{\sigma}\right)}{\Phi\left(\frac{\alpha - \mu}{\sigma}\right) - \Phi\left(\frac{\beta - \mu}{\sigma}\right)} \tag{a.3}$$

**i) VaR**

For expository purpose, let us omit the time index and denote $\delta p \equiv p_{1,t+1} - p_{1,t}$, $\delta L \equiv \delta L_{1,t+1} + \delta L_{0,t+1}$, $x_1 \equiv x_{1,t}$, $x_0 \equiv x_{0,t}$ and $VaR \equiv VaR_t^{\alpha}(\gamma) + Y_t$. The VaR is defined by:

$$\mathbb{P}\left[x_1 \delta p - \gamma(-x_0 - \delta L)^+ < -VaR\right] = \alpha, \tag{a.4}$$

where the time index is omitted in the conditioning. To compute the VaR, we first integrate out $\delta p$ given $\delta L$, then we integrate out $\delta L$. The conditional distribution of $\delta p$ given $\delta L$ is Gaussian with mean $\mu_p + \rho \frac{\sigma_P}{\sigma_L}(\delta L - \mu_L)$ and variance $\sigma_p^2(1-\rho^2)$. We deduce from Lemma i):

$$\begin{aligned}
\alpha &= \mathbb{E}\left\{\mathbb{P}\left[x_1 \delta p - \gamma(-x_0 - \delta L)^+ < -VaR | \delta L\right]\right\} && \text{(a.5)}\\
&= \mathbb{E}\left\{\mathbb{P}\left[\delta p < \frac{\gamma(-x_0 - \delta L)^+ - VaR}{x_1} | \delta L\right]\right\} && \text{(a.6)}\\
&= \mathbb{E}\left\{\Phi\left(\frac{\gamma(-x_0 - \mu_L - \sigma_L U)^+ - VaR - x_1\left(\mu_p + \rho\sigma_P U\right)}{x_1 \sigma_p \sqrt{1-\rho^2}}\right)\right\}, && \text{(a.7)}
\end{aligned}$$

where $U$ is a standard normal variable.

When $\gamma = 0$, the formula is greatly simplified. We have:

$$\begin{aligned}
\alpha &= \mathbb{P}\left[x_1 \delta p < -VaR\right] && \text{(a.8)}\\
&= \Phi\left(\frac{-VaR/x_1 - \mu_p}{\sigma_p}\right) && \text{(a.9)}
\end{aligned}$$

or equivalently:

$$VaR = -x_1\left(\mu_p + \sigma_p \Phi^{-1}(\alpha)\right) \tag{a.10}$$

**ii)** *PU*

Let us omit the time index and denote $\delta L \equiv \delta L_{1,t+1} + \delta L_{0,t+1}$. Since $\delta L \sim \mathcal{N}(\mu_L, \sigma_L^2)$, Lemma i) gives:

$$PU = \mathbb{P}\left[\delta L < -x_0\right] = \Phi\left(\frac{-x_0 - \mu_L}{\sigma_L}\right) \tag{a.11}$$

**iii)** *EUGU*

We deduce from Lemma ii):

$$\frac{EUGU_t(\gamma)}{1+\gamma} = -x_0 - \mathbb{E}\left[\delta L | \delta L < -x_0\right] = -x_0 - \mu_L + \sigma_L \frac{\varphi(\frac{-x_0-\mu_L}{\sigma_L})}{\Phi(\frac{-x_0-\mu_L}{\sigma_L})}, \tag{a.12}$$

$$EUGU_t(\gamma) = -(1+\gamma)\left(x_{0,t} + \mu_L - \sigma_L \frac{\varphi\left(\dfrac{-x_{0,t}-\mu_L}{\sigma_L}\right)}{\Phi\left(\dfrac{-x_{0,t}-\mu_L}{\sigma_L}\right)}\right). \tag{a.13}$$

# 3 Appendix: VaR Sensitivity

Lemma 1 in Gouriéroux, Laurent and Scaillet (2000) is:
*Let us consider a bivariate continuous vector* $(\boldsymbol{X}, \boldsymbol{Y})$ *and the quantile* $Q(\varepsilon, \alpha)$ *defined by*

$$\mathbb{P}\left[\mathbf{X} + \varepsilon \mathbf{Y} > Q(\varepsilon, \alpha)\right] = \alpha.$$

*Then:*

$$\frac{\partial}{\partial \varepsilon} Q(\varepsilon, \alpha) \mathbb{E}\left[\mathbf{Y} | \mathbf{X} + \varepsilon \mathbf{Y} = Q(\varepsilon, \alpha)\right].$$

We cannot apply directly this Lemma since in our case the quantile $Q(\varepsilon, \alpha)$ is defined by:

$$\mathbb{P}\left[\mathbf{X} + \varepsilon \mathbf{Y}^{+} > Q(\varepsilon, \alpha)\right] = \alpha. \tag{a.14}$$

However, the proof of Lemma 1 can be adapted for this case. Denoting by $f(x, y)$ the joint p.d.f. of the pair $(X, Y)$, we get:

$$\mathbb{P}\left[\mathbf{X} + \varepsilon \mathbf{Y}^{+} > Q(\varepsilon, \alpha)\right] = \alpha \Longleftrightarrow \int \left[ \int_{Q(\varepsilon,\alpha) - \varepsilon y^{+}} f(x, y) dx \right] dy = \alpha \tag{a.15}$$

$$\Longleftrightarrow \int_{y>0} \left[ \int_{Q(\varepsilon,\alpha) - \varepsilon y} f(x, y) dx \right] dy + \int_{y<0} \left[ \int_{Q(\varepsilon,\alpha)} f(x, y) dx \right] dy = \alpha. \tag{a.16}$$

Let us now differentiate with respect to $\varepsilon$:

$$\int_{y>0} \left( \frac{\partial Q(\varepsilon, \alpha)}{\partial \varepsilon} - y \right) f(Q(\varepsilon, \alpha) - \varepsilon y, y) dy + \int_{y<0} \frac{\partial Q(\varepsilon, \alpha)}{\partial \varepsilon} f(Q(\varepsilon, \alpha), y) dy = 0 \tag{a.17}$$

$$\int \frac{\partial Q(\varepsilon, \alpha)}{\partial \varepsilon} f(Q(\varepsilon, \alpha) - \varepsilon y^{+}, y) dy - \int_{y>0} y f(Q(\varepsilon, \alpha) - \varepsilon y, y) dy = 0 \tag{a.18}$$

$$\frac{\partial Q(\varepsilon, \alpha)}{\partial \varepsilon} = \frac{\displaystyle\int_{y>0} y f(Q(\varepsilon, \alpha) - \varepsilon y, y) dy}{\displaystyle\int f(Q(\varepsilon, \alpha) - \varepsilon y^{+}, y) dy} \tag{a.19}$$

$$\frac{\partial Q(\varepsilon, \alpha)}{\partial \varepsilon} = \frac{\displaystyle\int y^{+} f(Q(\varepsilon, \alpha) - \varepsilon y, y) dy}{\displaystyle\int f(Q(\varepsilon, \alpha) - \varepsilon y^{+}, y) dy} \tag{a.20}$$

$$\frac{\partial Q(\varepsilon, \alpha)}{\partial \varepsilon} = \frac{\mathbb{E}\left[Y^{+} | X + \varepsilon Y = Q(\varepsilon, \alpha)\right] \times \mathbb{P}\left[X + \varepsilon Y = Q(\varepsilon, \alpha)\right]}{\mathbb{P}\left[X + \varepsilon Y^{+} = Q(\varepsilon, \alpha)\right]} \tag{a.21}$$

$$\frac{\partial Q(\varepsilon, \alpha)}{\partial \varepsilon} = \mathbb{E}\left[Y^{+} | X + \varepsilon Y = Q(\varepsilon, \alpha)\right] \times \frac{\mathbb{P}\left[X + \varepsilon Y = Q(\varepsilon, \alpha)\right]}{\mathbb{P}\left[X + \varepsilon Y^{+} = Q(\varepsilon, \alpha)\right]}. \tag{a.22}$$

In our framework, with the notation used in Appendix 2, we have:

$$\mathbb{P}\left[ \underbrace{-Y_t - x_1 \delta p}_{\mathbf{X}} + \underbrace{\gamma}_{\varepsilon} \underbrace{(-x_0 - \delta L)^{+}}_{\mathbf{Y}} > \underbrace{VaR_t(\alpha, \gamma)}_{Q(\varepsilon,\alpha)} \right] = \alpha. \tag{a.23}$$

Therefore we get:

$$\begin{aligned}\frac{\partial VaR_t^\alpha(\gamma)}{\partial\gamma} =&\mathbb{E}\left[(-x_0-\delta L)^+ | -Y_t - x_1\delta p + \gamma(-x_0-\delta L) = VaR_t^\alpha(\gamma)\right] \\ &\times \frac{\mathbb{P}\left[-Y_t - x_1\delta p + \gamma(-x_0-\delta L) = VaR_t^\alpha(\gamma)\right]}{\mathbb{P}\left[-Y_t - x_1\delta p + \gamma(-x_0-\delta L)^+ = VaR_t^\alpha(\gamma)\right]}.\end{aligned} \tag{a.24}$$

When $\gamma = 0$, it reduces to:

$$\frac{\partial VaR_t^\alpha(0)}{\partial\gamma} = \mathbb{E}\left[(-x_0-\delta L)^+ | -Y_t - x_1\delta p = VaR_t^\alpha(0)\right] \times \frac{\mathbb{P}\left[-Y_t - x_1\delta p = VaR_t^\alpha(0)\right]}{\mathbb{P}\left[-Y_t - x_1\delta p = VaR_t^\alpha(0)\right]} \tag{a.25}$$

$$\frac{\partial VaR_t^\alpha(0)}{\partial\gamma} = \mathbb{E}\left[(-x_0-\delta L)^+ | -Y_t - x_1\delta p = VaR_t^\alpha(0)\right]. \tag{a.26}$$

# 4 Appendix: Summary statistics in the Gaussian case

As in the case of a limited credit line, let us consider Gaussian shocks:

$$\begin{pmatrix} \delta p_{1,t+1} \\ \delta L_{1,t+1} + \delta L_{0,t+1} \end{pmatrix} \sim \mathcal{N}\left( \begin{pmatrix} \mu_p \\ \mu_L \end{pmatrix} ; \begin{pmatrix} \sigma_p^2 & \rho\sigma_p\sigma_L \\ \rho\sigma_p\sigma_L & \sigma_L^2 \end{pmatrix} \right). \tag{a.27}$$

## 4.1 Value-at-Risk

As usual, the computation are based on $\tilde{Y}^*_{t+1|t}$ and not on $Y^*_{t+1|t}$: the expressions do not take into account the impossibility to sell more illiquid asset than the available one.
The expression of the VaR is obtained along the same lines as in Appendix 2.i):

$$\begin{aligned}
\alpha &= \mathbb{P}\left[x_1\delta p - \gamma min\left(M, (-x_0 - \delta L)^+\right) - \frac{1}{H}(-x_0 - \delta L - M)^+ < -VaR\right] \\
\alpha &= \mathbb{E}\left\{\mathbb{P}\left[x_1\delta p - \gamma min\left(M, (-x_0 - \delta L)^+\right) - \frac{1}{H}(-x_0 - \delta L - M)^+ < -VaR \middle| \delta L\right]\right\} \\
\alpha &= \mathbb{E}\left\{\mathbb{P}\left[\delta p < \frac{\gamma min\left(M, (-x_0 - \delta L)^+\right) + \frac{1}{H}(-x_0 - \delta L - M)^+ - VaR}{x_1} \middle| \delta L\right]\right\} \\
\alpha &= \mathbb{E}\left\{\Phi\left(\frac{1}{\sigma_p\sqrt{1-\rho^2}}\left(\frac{\gamma min\left(M, (-x_0 - \delta L)^+\right) + \frac{1}{H}(-x_0 - \delta L - M)^+ - VaR}{x_1}\right)\right.\right. \\
&\qquad \left.\left. -\mu_p - \rho\frac{\sigma_p}{\sigma_L}(\delta L - \mu_L)\right)\right\} \\
\alpha &= \mathbb{E}\left\{\Phi\left(\frac{\gamma min\left(M, (-x_0 - \mu_L - \sigma_L U)^+\right) + \frac{1}{H}(-x_0 - \mu_L - \sigma_L U - M)^+ - VaR - \rho\sigma_p U}{x_1\sigma_p\sqrt{1-\rho^2}}\right)\right\},
\end{aligned} \tag{a.28}$$

where $U$ is a standard Gaussian variable.
Indeed, when $M = \infty$, the expression reduces to the one in case of unlimited credit line (see Eq. 23).

## 4.2 Use of the credit line

Let us now focus on the need for cash. We get the probability of use (which corresponds to the probability of a liquidity incident) with Lemma i):

$$PU_t = \Phi\left(\frac{-x_{0,t} - \mu_L}{\sigma_L}\right). \tag{a.29}$$

To get the expected use given use is, we apply Lemma iii):

$$\frac{EUGU}{1+\gamma} = \mathbb{E}\left[-x_0 - \delta L | 0 < -x_0 - \delta L < M\right] \tag{a.30}$$

$$= -x_0 - \mathbb{E}\left[\delta L | -x_0 - M < \delta L < -x_0\right] \tag{a.31}$$

$$= -x_0 - \mu_L - \sigma_L \frac{\varphi\left(\frac{-x_0-M-\mu_L}{\sigma_L}\right) - \varphi\left(\frac{-x_0-\mu_L}{\sigma_L}\right)}{\Phi\left(\frac{-x_0-\mu_L}{\sigma_L}\right) - \Phi\left(\frac{-x_0-M-\mu_L}{\sigma_L}\right)}. \tag{a.32}$$

$$EUGU = -(1+\gamma)\left(x_0 + \mu_L + \sigma_L \frac{\varphi\left(\frac{-x_0-M-\mu_L}{\sigma_L}\right) - \varphi\left(\frac{-x_0-\mu_L}{\sigma_L}\right)}{\Phi\left(\frac{-x_0-\mu_L}{\sigma_L}\right) - \Phi\left(\frac{-x_0-M-\mu_L}{\sigma_L}\right)}\right). \tag{a.33}$$

## 4.3 Sell illiquid asset

Lemma i) gives:

$$PS_t = \mathbb{P}\left[x_0 + \delta L < -M\right] = \Phi\left(\frac{-M + x_0 - \mu_L}{\sigma_L}\right). \tag{a.34}$$

The probability of selling illiquid asset is:

$$PS_t = \Phi\left(\frac{-M + x_{0,t} - \mu_L}{\sigma_L}\right). \tag{a.35}$$

The expected volume of sell of illiquid assets is given by Lemma ii):

$$H \times ESGS = \mathbb{E}\left[-x_0 - M - \delta L | -x_1 p_{1,t+1} H < -x_0 - M + \delta L < 0\right] \tag{a.36}$$

$$= -x_0 - M - \tilde{\mu}_L - \tilde{\sigma}_L \frac{\varphi\left(\frac{-x_0-M-\tilde{\mu}_L-x_1 p_{1,t+1} H}{\tilde{\sigma}_L}\right) - \varphi\left(\frac{-x_0-M-\tilde{\mu}_L}{\tilde{\sigma}_L}\right)}{\Phi\left(\frac{-x_0-M-\tilde{\mu}_L}{\tilde{\sigma}_L}\right) - \Phi\left(\frac{-x_0-M-\tilde{\mu}_L-x_1 p_{1,t+1} H}{\tilde{\sigma}_L}\right)} \tag{a.37}$$

with $\tilde{\sigma}_L = \sqrt{1-\rho^2}\sigma_L$ and $\tilde{\mu}_L = \mu_L + \rho\frac{\sigma_L}{\sigma_p}(p_{1,t+1} - p_{1,t} - \mu_p)$.
Or if $p_{1,t+1}$ is unknown:

$$H \times ESGS = -x_0 - M - \mu_L - \mathbb{E}\Bigg\{\rho\sigma_p U - \sqrt{1-\rho^2}\sigma_L \times \frac{\varphi\left(\frac{-x_0-M-\mu_L-\rho\sigma_p U - x_1(p_{1,t}+\mu_p+\sigma_p U)H}{\sqrt{1-\rho^2}\sigma_L}\right) - \varphi\left(\frac{-x_0-M-\mu_L-\rho\sigma_p U}{\sqrt{1-\rho^2}\sigma_L}\right)}{\Phi\left(\frac{-x_0-M-\mu_L-\rho\sigma_p U}{\sqrt{1-\rho^2}\sigma_L}\right) - \Phi\left(\frac{-x_0-M-\mu_L-\rho\sigma_p U - x_1(p_{1,t}+\mu_p+\sigma_p U)H}{\sqrt{1-\rho^2}\sigma_L}\right)}\Bigg\}, \tag{a.38}$$

where $U$ is a standard Gaussian variable.

## 4.4 Bankruptcy

For the probability of bankruptcy, Lemma i) gives:

$$PB = \mathbb{P}\left[\delta L < -\left(x_1 p_{1,t+1} H + x_0 + M\right)\right] = \Phi\left(\frac{-\left(x_1 p_{1,t+1} H + x_0 + M\right) - \mu_L}{\sigma_L}\right). \tag{a.39}$$

Or if $p_{1,t+1}$ is unknown:

$$\begin{aligned}
PB &= \mathbb{P}\left[\delta L < -\left(x_1 p_{1,t+1} H + x_0 + M\right)\right] && \text{(a.40)}\\
&= \mathbb{P}\left[\delta L + x_1 \delta p H < -\left(x_1 p_{1,t} H + x_0 + M\right)\right] && \text{(a.41)}\\
&= \Phi\left(\frac{-\left(x_1 p_{1,t} H + x_0 + M\right) - \mu_L - \mu_p H x_1}{\sqrt{\sigma_L^2 + H^2 x_1^2 \sigma_p^2 + 2\rho H x_1 \sigma_p \sigma_L}}\right) && \text{(a.42)}\\
&= \Phi\left(\frac{-\left(x_1 (p_{1,t} + \mu_p) H + x_0 + M + \mu_L\right)}{\sqrt{\sigma_L^2 + H^2 x_1^2 \sigma_p^2 + 2\rho H x_1 \sigma_p \sigma_L}}\right). && \text{(a.43)}
\end{aligned}$$

# 5 Appendix: The geometry of regimes

## 5.1 The figure

The 8 potential regimes are represented in Figure 7.
Let us define:

- Point $A$. At point $A$, the financial institution is on the hedge of insolvency and the complete credit line is exactly sufficient to cover its liquidity needs. Therefore, $Y^*_{t+1|t}(A) = 0$ and $\delta L_{t+1}(A) = -x_{0,t} - \tilde{M}$.
- Point $B$. At point $B$, the financial institution has no more illiquid asset to sell and the complete credit line is exactly sufficient to cover its liquidity needs. Therefore, $-x_{0,t} - \delta L_{t+1}(B) - \tilde{M} = x_{1,t}(p_{1,t} + \delta p_{1,t+1}(B))H$ and $\delta L_{t+1}(B) = -x_{0,t} - \tilde{M}$.
- Point $C$. At point $C$, the financial institution is on the hedge of insolvency and has no more illiquid asset to sell. Therefore, $Y^*_{t+1|t}(C) = 0$ and $-x_{0,t} - \delta L_{t+1}(C) - \tilde{M} = x_{1,t}(p_{1,t} + \delta p_{1,t+1}(C))H$.
- Point $E$. At point $E$, the financial institution is on the hedge of insolvency and covers its liquidity needs by using exactly its cash. Therefore $Y^*_{t+1|t}(E) = 0$ and $\delta L_{t+1}(C) = -x_{0,t}$.

The coordinates of these points are:

$$x_A = -x_{0,t} - \tilde{M} \quad ; \quad y_A = -\frac{Y_t}{x_{1,t}} + \gamma \frac{\tilde{M}}{x_{1,t}} \tag{a.44}$$

$$x_B = -x_{0,t} - \tilde{M} \quad ; \quad y_B = -p_{1,t} \tag{a.45}$$

$$x_C = -L_{1,t} - L_{0,t} - (1+\gamma)\tilde{M} \quad ; \quad y_C = -p_{1,t} + \frac{1}{Hx_{1,t}}(-x_{0,t} - L_{1,t} - L_{0,t}) \tag{a.46}$$

$$x_E = -x_{0,t} \quad ; \quad y_E = -\frac{Y_t}{x_{1,t}} \tag{a.47}$$

## 5.2 Proof of Proposition 1

**i) Exclusion of regime $\mathcal{R}(D, A)$**
The abscissa of point $C$ is $x_C = -L_{1,t} - L_{0,t} - (1+\gamma)\tilde{M}$. $x_C$ is always smaller than $-L_{0,t}$, which is the limit of the shock on the liability side (see equation (5)). Thus point $C$ is not in the feasible set and regime $\mathcal{R}(D, A)$ cannot arise. Thus the computation above confirms the remark done in the main part of the text.

**ii) Liquidity regimes**
The assumption $L_{0,t} > x_{0,t} + \tilde{M}$ insures that the four liquidity regimes may arise in the feasible set. Equivalently, the assumption states that the vertical limit introduced by the constraint $\delta L_{t+1} \geq -L_{0,t}$ is between point $C$ and point $A$.

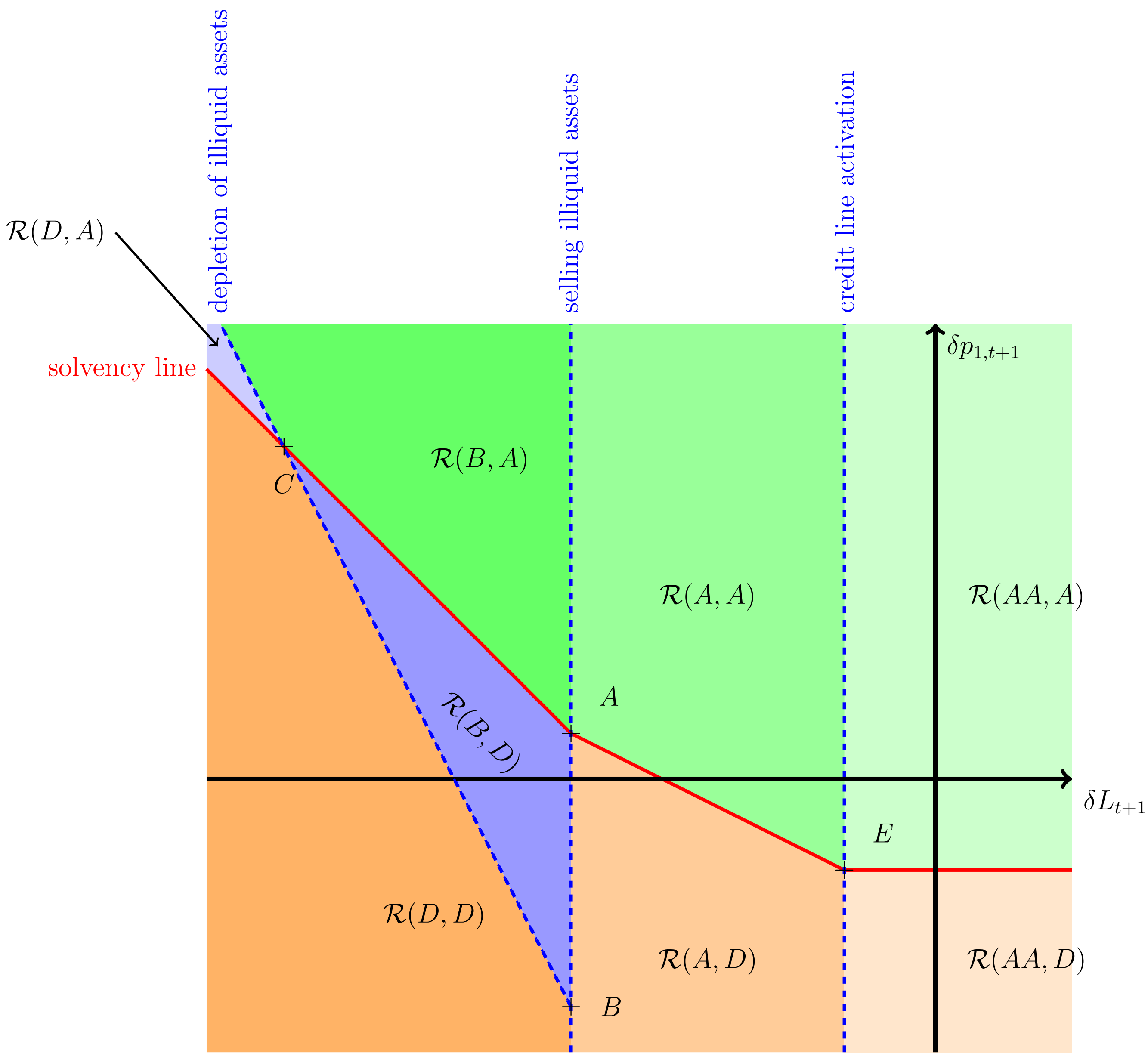


Figure 7: All Potential Regimes (case $B$ below $A$)

**iii) Solvency line**

The assumption $Y_t < x_{1,t}p_{1,t} + \gamma\tilde{M}$ insures point $B$ is below point $A$ . The solvency line is in the feasible set.

## 5.3 Case $Y_t > x_{1,t}p_{1,t} + \gamma\tilde{M}$

Considering $Y_t > x_{1,t}p_{1,t} + \gamma\tilde{M}$ implies that point $B$ is above point $A$. The situation is represented in Figure 8.
The abscissa of point $C$ is still smaller than he limit of the shock on the liability side. Consequently, regimes $\mathcal{R}(B, A)$ and $\mathcal{R}(D, D)$ cannot arise.

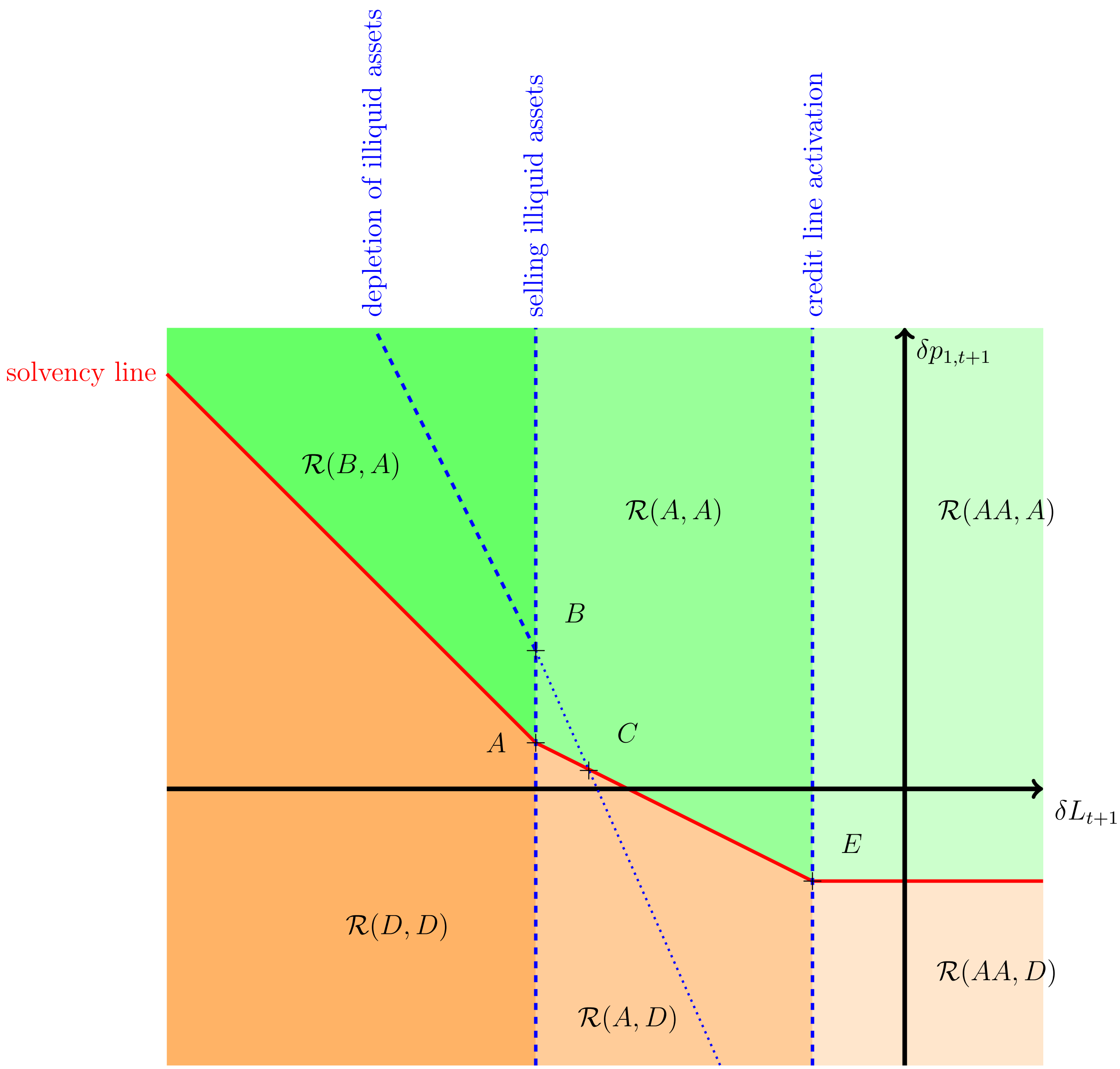


Figure 8: All Potential Regimes (case $B$ above $A$)

# 6 Appendix: The geometry of regimes with reserves

## 6.1 The figure

The 15 potential regimes are represented in Figure 9.

Let us define:

- Point $A$. At point $A$, the financial institution is on the hedge of insolvency and the complete credit line is exactly sufficient to cover its liquidity needs. Therefore, $Y^*_{t+1|t}(A) = 0$ and $\delta L_{t+1}(A) = -x_{0,t} - \tilde{M}$.
- Point $B$. At point $B$, the financial institution has no more illiquid asset to sell and the complete credit line is exactly sufficient to cover its liquidity needs. Therefore, $-x_{0,t} - \delta L_{t+1}(B) - \tilde{M} = x_{1,t}(p_{1,t} + \delta p_{1,t+1}(B))H$ and $\delta L_{t+1}(B) = -x_{0,t} - \tilde{M}$.
- Point $C$. At point $C$, the financial institution is on the hedge of insolvency and has no more illiquid asset to sell. Therefore, $Y^*_{t+1|t}(C) = 0$ and $-x_{0,t} - \delta L_{t+1}(C) - \tilde{M} = x_{1,t}(p_{1,t} + \delta p_{1,t+1}(C))H$.
- Point $E$. At point $E$, the financial institution is on the hedge of insolvency and covers its liquidity needs by using exactly its cash. Therefore $Y^*_{t+1|t}(E) = 0$ and $\delta L_{t+1}(C) = -x_{0,t}$.
- Point $G$. At point $G$, the financial institution is on the hedge of insolvency and the reserve $R_{1,t}$ has been completely used. Therefore, $Y^*_{t+1|t}(G) = 0$ and $\delta L_{t+1}(G) = -x_{0,t} - \tilde{M} - R_{1,t} - x_{1,t}H(p_{1,t} + \delta p_{1,t+1})$.
- Point $I$. At point $I$, the financial institution is exactly fulfilling capital regulation and has no more illiquid asset to sell. Therefore, $Y^*_{t+1|t}(I) = R_{2,t} + R_{2,t}$ and $-x_{0,t} - \delta L_{t+1}(I) - \tilde{M} = x_{1,t}(p_{1,t} + \delta p_{1,t+1}(I))H$.

With reserves, the PnL $Y^*_{t+1|t}$ becomes:

$$\begin{aligned} Y^*_{t+1|t} = \Big(& Y_t + x_{1,t}\delta p_{1,t+1} - \gamma min \left[\tilde{M}; (-x_{0,t} - \delta L_{1,t+1} - \delta L_{0,t+1})^+\right] \\ & - \left(\frac{1}{H} - 1\right) min \left[(-x_{0,t} - \delta L_{1,t+1} - \delta L_{0,t+1} - \tilde{M})^+; x_{1,t}H(p_{1,t} + \delta p_{1,t+1}\right]\Big) \\ \times\, & \mathbb{1}_{\left(-x_{0,t} - \delta L_{1,t+1} - \delta L_{0,t+1} - \tilde{M} - x_{1,t}H(p_{1,t} + \delta p_{1,t+1})\right)^+ < R_{1,t}}. \end{aligned} \tag{a.48}$$

The formula is very close to the one without reserve. The term corresponding to illiquid asset selling is now bounded by the quantity of available illiquid asset and the dummy variable corresponds to the complete use of the reserve. Note that when the reserve is used, there is no decrease of the PnL since the reserve is perfectly liquid (when used).

The coordinates of the aforementioned points are:

$$x_A = -x_{0,t} - \tilde{M} \quad ; \quad y_A = -\frac{Y_t}{x_{1,t}} + \gamma \frac{\tilde{M}}{x_{1,t}} \tag{a.49}$$

$$x_B = -x_{0,t} - \tilde{M} \quad ; \quad y_B = -p_{1,t} \tag{a.50}$$

$$x_C = -L_{1,t} - L_{0,t} - (1+\gamma)\tilde{M} \quad ; \quad y_C = -p_{1,t} + \frac{1}{Hx_{1,t}}\left(-x_{0,t} - L_{1,t} - L_{0,t}\right) \tag{a.51}$$

$$x_E = -x_{0,t} \quad ; \quad y_E = -\frac{Y_t}{x_{1,t}} \tag{a.52}$$

$$x_G = -L_{1,t} - L_{0,t} - (1+\gamma)\tilde{M} - R_{1,t} \quad ; \quad y_G = -\frac{\frac{1}{H} - 1}{H} p - \frac{Y_t}{x_{1,t}H} + \gamma \frac{\tilde{M}}{x_{1,t}H} + \frac{R_{1,t}}{x_{1,t}H} \tag{a.53}$$

$$x_I = -L_{1,t} - L_{0,t} - (1+\gamma)\tilde{M} - \frac{1}{H}\left(R_{1,t} + R_{2,t}\right)$$
$$; \quad y_I = -\frac{Y_t}{x_{1,t}H} + \gamma \frac{\tilde{M}}{x_{1,t}H} + \frac{R_{1,t} + R_{2,t}}{x_{1,t}H} + \left(\frac{1}{H} - 1\right) \tag{a.54}$$

## 6.2 Proof of Proposition 2

**i) Exclusion of Regimes $\mathcal{R}(C,C)$, $\mathcal{R}(C,A)$, $\mathcal{R}(D,A)$ and $\mathcal{R}(D,C)$**

The abscissa of point $C$ is $x_C = -L_{1,t} - L_{0,t} - (1+\gamma)\tilde{M}$. $x_C$ is always smaller than $-L_{0,t}$, which is the limit of the shock on the liability side (see equation (5)). Thus point $C$ is not in the feasible set. Moreover, the abscissas of point $G$ and $I$ are lower than the abscissa of point $C$, therefore points $G$ and $I$ are not in the feasible set. Therefore, Regimes $\mathcal{R}(C,C)$, $\mathcal{R}(C,A)$, $\mathcal{R}(D,A)$ and $\mathcal{R}(D,C)$ cannot arise.

**ii) Liquidity regimes** The assumption $L_{0,t} > x_0 + \tilde{M}$ insures that the four liquidity regimes may arise in the feasible set. Equivalently, the assumption states that the vertical limit introduced by the constraint $\delta L_{t+1} \geq -L_{0,t}$ is between point $C$ and point $A$.

**iii) Solvency regimes** The assumption $Y_t < x_{1,t}p_{1,t} + \gamma\tilde{M}$ and insures point $B$ is below point $A$ . The solvency line is in the feasible set.

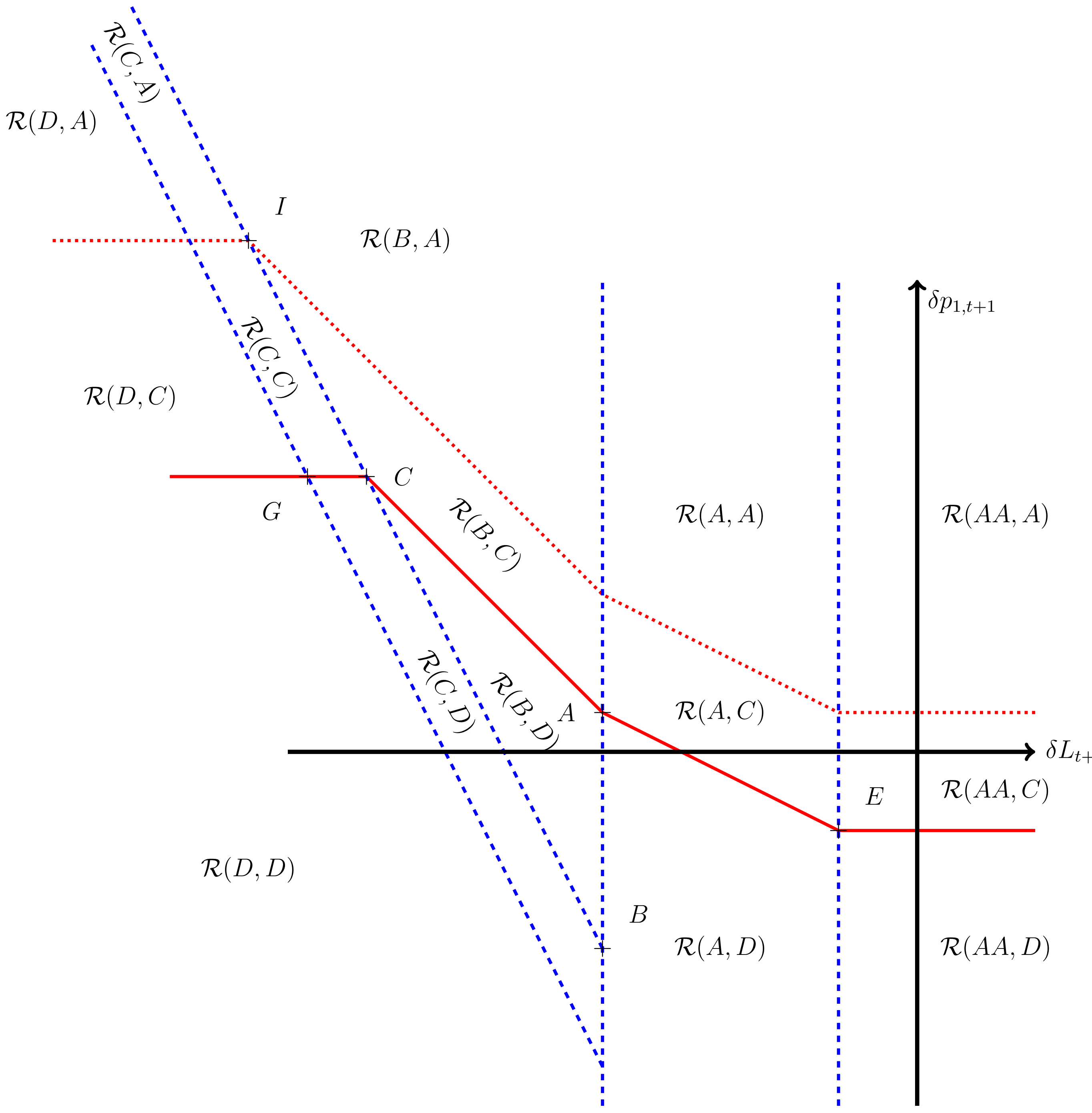


Figure 9: All Potential Regimes with Reserves (case $B$ below $A$)