Incentive Constrained Risk Sharing, Segmentation, and Asset Pricing*

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Abstract

We analyse a one-period general equilibrium asset pricing model with standard corporate finance frictions (cash-diversion). Incentive compatibility constraints imply that the market is endogenous incomplete. They also induce endogenous segmentation, as different types of investors hold different assets in equilibrium, and co-movements in asset prices. Equilibrium expected excess returns reflect two premia: a risk premium, which is positive if the return on the asset is large when the pricing kernel is low, but which does not reflect aggregate or individual consumption due to incentive compatibility constraints; and a divertibility premium, which is positive if the return on the asset large when incentive-compatibility constraints bind. This divertibility premium is inverse-U shaped with betas, in line with the empirical findings that the security market line is flat at the top.

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1 Introduction

Financial markets facilitate risk sharing. They allow agents to unwind their excess risk-exposure, and to buy and sell insurance from one another. For example, agents can sell credit default swaps (CDS), put options or other derivatives, such as futures. After the initiation of derivative positions, underlying asset values fluctuate, affecting the profitability of these positions. For example, after an agent sold puts against the occurrence of bad macro-states, if the likelihood of a recession increases, the expected liability of this agent increases as well. When the liabilities become large, the agent can be tempted to strategically default. To mitigate such default incentives, the agent's promises are backed by collateral assets.

Asset pricing in the presence of default incentives has been studied by Kehoe and Levine (1993, 2001), Alvarez and Jermann (2000), Chien and Lustig (2009) and Gottardi and Kubler (2015). These papers assume that tradeable assets and their payoffs are perfectly pledgeable, while other sources of income, such as labor income, are not tradable and cannot be seized when the agents default on their obligations. In contrast, corporate finance and financial intermediation theory emphasizes the payoffs of tradeable assets can be imperfectly pledgeable due to variety informational problems, notably ex-ante moral-hazard, as in Holmstrom and Tirole (1997), and ex-post moral-harzard, as in DeMarzo and Sannikov (2006) and DeMarzo and Fishman (2007), in line with Bolton and Scharfstein (1990).[1]

The contribution of this paper is to study how ex-post moral hazard, limiting the pledgeability of the payoff of tradeable assets, affects the completeness of the market, the pricing of tradeable assets, and their allocation across agents. In line with Kehoe and Levine (2001), Alvarez and Jermann (2000) and Chien and Lustig (2009), we show that incentive compatibility constraints create endogenous market incompleteness. Relative to this

[1]Suppose for example that the agent who sold the CDS is a hedge fund. In that case, assets can correspond to a dynamic trading strategy, possibly in opaque and illiquid markets. Effort then is necessary to minimize transactions costs, accurately estimate risk exposure and hedges, and monitor broker dealers. Effort is costly for the agent, but imperfectly observable by the counterparties, which implies that pledgeable income of the assets is lower than the total cash flow they generate. Similarly, suppose the agent who sold the CDS is an investment bank, who invested in a portfolio of loans. To ensure that these loans generate large payoffs, the investment bank must exert monitoring efforts, as in Holmstrom and Tirole (1997), to ensure that the firms receiving the loans use the resources efficiently. To the extent that effort is costly and unobservable there is a moral hazard problem, which implies that the pledgeable income of the assets held by the investment bank is lower than the total cash flow generated by its assets.

literature, we obtain new results concerning the asset pricing and allocation of tradeable assets.

First, we find that tradeable assets are priced below the corresponding replicating portfolio of Arrow securities. This does not generate arbitrage opportunities, however, because the price wedge reflects the shadow price of incentive compatibility constraints. In this context, equilibrium expected excess returns reflect two premia: a risk premium, which is positive if the return on the asset is large when the pricing kernel is low, but which does not reflect aggregate or individual consumption due to incentive compatibility constraints; and, a divertibility premium, which is positive if the return on the asset is large when incentive-compatibility constraints bind. This divertibility premium is inverse U shaped with betas, in line with the empirical findings that the security market line is flat at top.

Second, we find that the market for tradeable assets is endogenously segmented, as different types of agents hold different types of assets in equilibrium. This is because the equilibrium asset allocation optimally mitigates default incentives. Namely, agents who have large liabilities in a particular state of the world find it optimal to hold assets with low payoff in that state. We show that endogenous segmentation leads relatively risk-tolerant agents to hold riskier assets, and creates co-movement among the prices of assets held by the same clientele of agents.

We consider a canonical general equilibrium model. At time 0, competitive risk-averse agents are endowed with shares of real assets ("trees"), which they can trade, together with a complete set of Arrow securities. At time 1, the real assets generate consumption flows and agents consume. In this complete competitive market, if there were no friction, the first best would be attained in equilibrium. Risk would be shared perfectly, with less risk-averse agents insuring more risk-averse agents against adverse realizations of the aggregate state. The consumptions of all agents would comove with aggregate output. It is the risk associated with aggregate output that would determine the risk premium in the price of Arrow securities and real assets (see, e.g. Huang and Litzenberger (1988)). Finally, agents would be indifferent between holding a real asset and the corresponding portfolio of Arrow securities, since both would have the same arbitrage-free price. As a result, the allocation of real assets would be indeterminate.

We study how incentive constraints alter that outcome. To do so, we introduce the simplest possible incentive problem. At time 1, the agents who sold Arrow securities are supposed to transfer resources to the agents who bought these securities. Instead of delivering on their promises, these agents could strategically default and divert a fraction of the payoff of the assets they hold. Only the fraction of payoff that cannot be diverted is pledgeable, i.e., can be used to back the sale of Arrow securities. This is the sense in which collateral is imperfect, directly in line with the cash-diversion model of corporate finance (see DeMarzo and Fishman (2007) and DeMarzo and Sannikov (2006)). We show that, in equilibrium, the incentive compatibility constraints prevent relatively risk-tolerant agents from providing the first-best level of insurance to more risk-averse agents. Consequently, while there is a market for each Arrow security, the market is endogenously incomplete.

This framework delivers sharp novel implications for asset pricing and asset holdings. The prices of real assets ("trees") are equal to the value of their consumption flows, evaluated with the Arrow Debreu state prices, minus a "divertibility discount." The latter is the shadow price of the incentive constraint. Thus there is a form of underpricing, as the prices of real assets are lower than the prices of portfolios of Arrow securities generating the same consumption flows at time 1. This does not constitute an arbitrage opportunity, however. In order to conduct an arbitrage trade, an agent would need to sell Arrow securities and use the proceeds to buy assets. This is precluded by the incentive constraint: if the agent sold these Arrow securities, this would increase his liabilities, thus increasing his temptation to strategically default, and his incentive compatibility constraint would no longer hold. We also show that incentive compatibility constraints have implications for asset holdings. Namely, our model predicts that, to optimally mitigate incentive problems, agents should hold assets with low payoffs in the states against with they sell a large amount of Arrow securities. Thus, even if the cash diversion friction is constant across assets and agents, the market will be endogenously segmented: different agents will find it optimal to hold different types of assets in equilibrium. Unlike in models with exogenous segmentation, assets not only reflect the marginal utility of wealth of the asset holders, but also the shadow cost of their incentive constraints.

To further illustrate equilibrium properties, we consider the simple case in which there are two states, two agent's types, one more risk-tolerant and the other more-risk averse, and an arbitrary distribution of assets. In equilibrium, the risk-tolerant agent consumes relatively less in the bad than in the good state so as to insure the risk-averse agent. To implement this consumption allocation, the risk-tolerant agent sells Arrow securities that pay in the bad state, and so has more incentives to divert cash flow in the bad than in the good state. In equilibrium, these incentive problems are optimally mitigated if the risk-tolerant agent holds assets paying off much less in the low state than in the high state, that is, high beta assets. Within the set of high beta assets held by the risk-tolerant agent, the riskier ones, which have lower cash flow in the low state, create less incentive problems, have lower divertibility discounts and so are less under-priced. Symmetrically, the risk-averse agent hold low beta assets. Within the set of low beta assets, the safer ones also have lower divertibility discount and are less under-priced. This implies that the divertibility discount is inverse U shaped in beta, and that the security market line is flatter at the top, in line with Black (1972) and recent evidence by Frazzini and Pedersen (2014) and Hong and Sraer (2016). Another implication of this model is that a tightening of incentive problems creates co-movement in divertibility discounts. Suppose, for example, that some of the high-beta assets held by the risk-tolerant agents become more divertible. Then, the divertibility discount of these assets increases, and the divertibility discount of all the other assets held by the risk-tolerant agent increases by more than that of assets held by the risk-averse agent. Thus, co-movement in divertibility discount is stronger among assets held by the same type of agents.

Literature: Kehoe and Levine (1993, 2001), Alvarez and Jermann (2000), Chien and Lustig (2009) and Gottardi and Kubler (2015) have proposed dynamic models in which strategic default is deterred by exclusion from future markets, or by the loss of some perfectly pledgeable collateral. In our static model, by contrast, strategic default is deterred because cash flow diversion is inefficient and costly. But this is not the key ingredient at the root of the difference between their results and ours. The origin of the difference in results is that in their analysis human capital (generating labor income) is fully nonpledgeable, but not tradeable, while in our analysis all assets are tradeable but their cashflows are only partially pledgeable. This creates a wedge between the price of tradeable assets and that of the portfolio of Arrow securities, the divertibility discount, and it induces endogenous market segmentation.

The divertibility discount arising in our model may seem to contradict the conclusions of theoretical studies pointing towards a premium. For example, Fostel and Geanakoplos (2008) point to a "collateral premium", and Alvarez and Jermann (2000) notice that, under natural conditions, limited commitment frictions tend to increase asset prices. Similarly, new monetarist analyses point to a "liquidity premium" (see for example Lagos (2010), Li, Rocheteau, and Weill (2012), Lester, Postlewaite, and Wright (2012)). There is no contradiction, however, since our analysis also points to a premium. The difference is that the benchmark valuation is not the same for the premium and the discount results. The divertibility discount is the difference between the equilibrium price of a real asset and the price of a replicating portfolio of Arrow securities. There is also a premium, however, equal to the difference between the price of the asset and its value evaluated at the marginal utility of the agent holding it.

The next section presents the model. Section 3 presents general results on equilibrium and optimality. Section 4 presents more specific results, obtained when there are only two types of agents.

2 Model

2.1 Assets and Agents

There are two dates $t = 0, 1$. The state of the world ω realizes at $t = 1$ and is drawn from some finite set Ω according to the probability distribution $\{\pi(\omega)\}_{\omega\in\Omega}$, where $\pi(\omega) > 0$ for all ω. All real resources are the dividends of assets referred to as "trees." The set of tree types is taken to be a compact interval that we normalize to be $[0, 1]$, endowed with its Borel σ-algebra. The distribution of asset supplies is a positive and finite measure $\bar{N}$ over the set $[0, 1]$ of tree types. We place no restriction on $\bar{N}$: it can be discrete, continuous, or a mixture of both. The payoff of tree j in state $\omega \in \Omega$ is denoted by $d_j(\omega) \geq 0$, with at least one strict inequality in for some state $\omega \in \Omega$. An important technical condition for our existence proof is that, for all $\omega \in \Omega$, $j \mapsto d_j(\omega)$ is continuous. Economically, this means that trees are finely differentiated: nearby trees in $[0, 1]$ have nearby characteristics. But continuity is a mild assumption since we do not impose any restriction on the distribution of supplies.[2]

The economy is populated by finitely many types of agents, indexed by $i \in I$. The measure of type $i \in I$ agents is normalized to one. Agents of type $i \in I$ have Von Neumann Mortgenstern utility

$$U_i(c_i) \equiv \sum_{\omega\in\Omega} \pi(\omega) u_i \left[c_i(\omega)\right]$$

[2]For example, there could be a "hole" in the distribution of supplies, $\bar{N}([0, j_1]) = \bar{N}([0, j_2])$ for some $j_1 < j_2$. In that case j_1 and j_2 are adjacent in the support of $\bar{N}$, but they have discretely different characteristics.

over time $t = 1$ state-contingent consumption. We take the utility function to be either linear, $u_i(c) = c$, or strictly increasing, strictly concave, and twice-continuously differentiable over $c \in (0, \infty)$. Without loss of generality, we apply an affine transformation to the utility function $u_i(c)$ so that it satisfies either $u_i(0) = 0$; or $u_i(0) = -\infty$ and $u_i(\infty) = +\infty$; or $u_i(0) = -\infty$ and $u_i(\infty) = 0$. In addition, if $u_i(0) = -\infty$ we assume that there exists some $\gamma_i > 1$ such that, for all c small enough, $\frac{u_i'(c)c}{|u_i(c)|} \leq (\gamma_i - 1)$. This implies the Constant Relative Risk Aversion (CRRA) bound $0 \geq u_i(c) \geq Kc^{1-\gamma_i}$ for all c small enough and some negative constant K.

Finally, we assume that, at time $t = 0$, agent $i \in I$ is endowed a strictly positive share, $\bar{n}_i > 0$, in the market portfolio. Of course, agents' shares in the market portfolio must add up to one, that is $\sum_{i \in I} \bar{n}_i = 1$.

2.2 Markets, Budget Constraints, and Incentive Compatibility

Markets. At time zero, agents choose their holdings of trees, a finite measure over the set of tree types, $[0, 1]$. We assume that agents cannot short sell, i.e., they cannot own a negative fraction of a firm. Formally, agents must choose a portfolio of trees from the set $\mathcal{M}_+$ of *positive* finite measures over $[0, 1]$. Economically, the short-selling constraint restricts the set of state contingent payoffs attainable with tree portfolios only: this set is in general a strict subset of $\mathbb{R}_+^{|\Omega|}$, even when the collection of tree payoffs has rank $|\Omega|$.

In order to obtain state-contingent payoffs that are not attainable by tree portfolios, agents can buy or sell a complete set of Arrow securities. However, as will be explained below, incentive problems limit the quantity of Arrow securities an agents can sell. While the trees are in strictly positive net supply, Arrow securities are in zero net supply. The vector of agent i's positions in each of the Arrow securities is denoted by $a_i \equiv \{a_i(\omega)\}_{\omega \in \Omega}$. The position $a_i(\omega)$ can be positive (if the agent buys the Arrow security) or negative (if the agent sells the Arrow security).

Budget constraints. A *price system* for trees and Arrow securities is a pair (p, q), where $p : j \mapsto p_j$ is a continuous function for the price of tree j,[3] and $q = \{q(\omega)\}_{\omega \in \Omega}$ is a vector in $\mathbb{R}^{|\Omega|}$. Given the price system, the

[3]Hence, we assume that the price functional admits a dot-product representation based on a continuous function of tree type. This is a restriction: in full generality one should allow for any continuous linear functional, some of which do not have such representation. However, given our maintained assumption that trees are finely differentiated, this restriction turns out to be without loss of generality. Namely, one can show that *any* equilibrium allocation can be supported by a price functional represented by a continuous function of tree types. See the paragraph before Proposition 17 page 36.

time-zero budget constraint for agent i is:

$$\sum_{\omega\in\Omega} q(\omega)a_i(\omega) + \int p_j \, dN_{ij} \leq \bar{n}_i \int p_j \, d\bar{N}_j. \tag{1}$$

At time one, agent i's consumption must satisfy:

$$c_i(\omega) = a_i(\omega) + \int d_j(\omega) \, dN_{ij}. \tag{2}$$

We denote the state-contingent consumption plan by $c_i \equiv \{c_i(\omega)\}_{\omega\in\Omega}$.

Incentive compatibility. At time $t = 1$, the agent is supposed to follow the consumption plan given in (2). Instead, the agent could default on his contractual obligations, and divert the payoffs of the trees and Arrow securities he holds.

We assume specifically that the agent can divert a fraction $\delta \in [0, 1)$ of the dividend of tree j and of the Arrow security paying off in state $\omega \in \Omega$. Suppose for example that an agent of type i has a portfolio N_i of trees and a position $a_i(\omega)$ in the state ω Arrow security. If he chooses to divert in state ω, he runs away with fractions δ of the trees' payoff and of the payoff of the Arrow security. Hence, if he diverts, the agent consumes:

$$\hat{c}_i(\omega) = \delta \int d_j(\omega) \, dN_{ij} + \delta \max\{a_i(\omega), 0\}. \tag{3}$$

The incentive compatibility condition is such that the agent prefers not to default and divert, which is the case if

$$c_i(\omega) \geq \hat{c}_i(\omega),$$

where $c_i(\omega)$ is given in (2) and $\hat{c}_i(\omega)$ in (3). The incentive constraint can be rewritten as

$$(1-\delta) \int d_j(\omega) \, dN_{ij} + a_i(\omega) \geq \delta \max\{a_i(\omega), 0\}.$$

When $a_i(\omega) > 0$, since $\delta < 1$, this condition is slack, while, when $a_i(\omega) < 0$ it may bind and writes as

$$(1-\delta) \int d_j(\omega) \, dN_{ij} \geq -a_i(\omega). \tag{4}$$

The right–hand–side is the liability of the agent in state ω. The left-hand-side is the non-divertible part of the total state–ω output from the assets held by the agent. Thus, the incentive compatibility condition binds if the agent's state–ω liability is large (in that state the agent is very indebted) and the state–ω output from the assets of the agent is low.

This discussion implies that only the diversion of dividends, and not that of Arrow security payoffs, is relevant for the incentive compatibility constraint. Thus, using (2), the incentive compatibility condition simplifies to

$$c_i(\omega) \geq \delta \int d_j(\omega)\, dN_{ij}, \tag{5}$$

for all $\omega \in \Omega$, where the left-hand side is the consumption plan of the agent, and the right-hand side is what he would get if he were to divert.

2.3 Discussion

2.3.1 Interpreting the incentive compatibility constraint

If we define the equity capital of the agent in state ω as the difference between the output from his assets and its liabilities, the incentive compatibility constraint can be interpreted in terms of state-contingent capital requirements: equity capital must be large enough so that the agent is not tempted to strategically default.

Another interpretation of the constraint is in terms of haircuts. How much the agent can promise to pay in state ω is an increasing function of his output in that state. That output serves as collateral for the payment promised by the agent. But, as can be seen in (4), the amount the agent can promise is lower than the face value of the collateral, because some of that collateral could be diverted. The wedge between the output/collateral and the maximum promised payment can be interpreted as a haircut. Haircuts are increasing in δ. Haircuts are not imposed on an individual asset basis, but at the level of the aggregate position, or portfolio of the agent. This is in line with the practice of "portfolio margining."

Note that the capital requirement, or haircut, is not imposed by the regulator. It is requested by the private contracting agents to limit counterparty risk. There is however an aspect of that requirement that cannot be completely decentralized. The incentive compatibility constraint of agent i involves the Arrow securities traded by agent i with all other agents in the economy. These multiple trades must be aggregated (and cleared) to determine the total exposure of agent i to state ω, and then compared to the assets of the agent, imputing the right haircuts. This can be the role of the Central Clearing Party (CCP), which in our model can centralize and clear all trades to ensure incentive compatibility, and thus deliver a better outcome than the outcome which would arise with bilateral contracting only. For example, if agent i has already sold an amount

$$-a_i(\omega) = (1-\delta)\int d_j(\omega)\, dN_{ij}$$

of state–ω Arrow security to agents i' and i'' (so that (4) binds). Then agent i should not be allowed by the CCP to sell an additional amount of that security to agent i'''. In a completely decentralized market, with bilateral contracting only, such a deviation could be tempting, depending on the bankruptcy rules.[4] With CCP centralized clearing ensuring that the incentive compatibility constraint holds, there is no need to specify bankruptcy rules, since bankruptcy never occurs.

2.3.2 Tradeability and divertibility

The agents in our model could be financial institutions, e.g. banks making loans to firms, or venture capitalists holding stakes in innovative projects. In such context, $d_j(\omega)$ is the payoff generated by firm or project j in state ω. To ensure that this payoff is actually generated, and available to pay the liability $a_i(\omega) < 0$ of i, the agent must monitor the project, which takes effort, time and resources. If this effort is not incurred, the project only delivers $(1-\delta)d_j(\omega)$, instead of $d_j(\omega)$.[5] Thus, $\delta\, d_j(\omega)$ can be interpreted as the opportunity cost of effort. This is very similar to the classical moral hazard problem of unobservable effort of Holmstrom and Tirole (1997). In their analysis the moral hazard problem is formulated in terms of private benefits, instead of cost of effort. Similarly, in our analysis, δ can be interpreted in terms of private benefit. The main difference here is that effort takes place after the state ω is realized, so we consider ex-post moral hazard, while Holmstrom and Tirole (1997) consider ex-ante moral hazard.

Instead of investments in non financial firms, assets could be made of illiquid securities, or investment strategies in Over the Counter (OTC) markets - not explicitly modeled in the present paper. In that context diversion can be interpreted as failing to take the appropriate actions maximizing the value of the investment. For example, this can involve failing to incur the cost of effort necessary to minimize transactions costs. Or it

[4] Attar, Mariotti, and Salanié (2011, 2014) analyse the problems arising when agents trade in market with non exclusivity. Their setting differs from ours, however, in particular because they consider adverse selection.

[5] What does it mean that the set of type j loans is divided amongst many agents? All the loans in that set are to similar firms in the same sector. That set is then split in smaller subsets held by a different financial institution.

could involve selling at a really good price to another institution, in exchange for kick backs. Or it could involve letting an intermediary front run, again in exchange for kick backs.

There is yet another possible microfoundation for our divertibility assumption. Suppose that, if agent i defaults, his creditors can only get a fraction $1-\delta$ of the cash flows from the assets of i. Furthermore, suppose that at time 1, in state ω, before payments are made, agent i can make a take-it-or-leave-it offer to his creditor: "Either you scale down the total amount of my state-ω contingent debt to $(1-\delta)\int d_j(\omega)\, dN_{ij}$ or I default." Knowing that, if agent i defaults, she gets no more than $(1-\delta)\int d_j(\omega) dN_{ij}$, the creditor accepts agent i's offer. Hence the incentive compatibility condition we postulate.

In the main body of the paper we assume for simplicity that δ is constant across agents and assets. In the appendix all our proofs cover the generalized case in which the divertibility parameter is a continous function function δ_{ij} of the identity i of the agent and of the type j of the asset. This may be a natural assumption to make in some contexts. For example, it can be that different agents have different costs of monitoring effort, or different costs of transactions costs reduction. It can also be that different agents have different networking capital, and correspondingly different abilities to obtain kick backs. It can also be that different institutions are more or less transparent, and that transparency reduces the ability to engage in the above discussed diversion activities. Likewise, assets traded in transparent and liquid markets will offer less opportunities for dealing at inflated or deflated prices and also less opportunities for front running. Also, simple standard assets will be less divertible. In contrast, complex non standard assets, traded in opaque, illiquid OTC markets will offer more opportunities for diversion.

3 Equilibrium, arbitrage and optimality

3.1 The agent's problem

As is standard one can consolidate the time-zero and the time-one budget constraints into a single inter-temporal budget constraint. That is, the state-contingent consumption plan c_i and the tree holdings N_i satisfy the time-zero budget constraint (1) and the time-one budget constraint (2), if and only if

$$\sum_{\omega\in\Omega} q(\omega)c_i(\omega) + \int p_j\, dN_{ij} \leq \bar{n}_i \int p_j\, d\bar{N}_j + \sum_{\omega\in\Omega} q(\omega) \int d_j(\omega)\, dN_{ij}. \tag{6}$$

Notice that both the budget constraint (6) and the incentive compatibility constraint (5) are only a function of (c_i, N_i), and do not depend on the Arrow security holdings a_i. Hence, as is standard, we define the consumption set of agent $i \in I$ to be $X_i \equiv \mathbb{R}_+^{|\Omega|} \times \mathcal{M}_+$, the product of the set of positive state contingent consumption plans and of the set of positive finite measures over tree types.

The problem of agent i is, then, to maximize $U_i(c_i)$ with respect to $(c_i, N_i) \in X_i$, subject to the the intertemporal budget constraint (6) and the incentive compatibility condition (5).

3.2 Definition of Equilibrium

Let X denote the cartesian product of all agents' consumption set. An *allocation* is a collection $(c, N) = (c_i, N_i)_{i \in I} \in X$ of consumption plans and tree holdings for every agent $i \in I$. An allocation (c, N) is *feasible* if it satisfies:

$$\sum_{i \in I} c_i(\omega) \leq \sum_{i \in I} \int d_j(\omega)\, dN_{ij} \text{ for all } \omega \in \Omega \tag{7}$$

$$\sum_{i \in I} N_i = \bar{N}. \tag{8}$$

An *equilibrium* is a feasible allocation (c, N) and a price system (p, q) such that, for all $i \in I$, (c_i, N_i) solves agent's i problem given prices.

3.3 Some elementary properties of equilibrium

3.3.1 Incentive-Constrained Pareto Optimality

An allocation $(c, N) \in X$ is said to be *incentive-feasible* if it satisfies the incentive compatibility constraints (5) for all $(i, \omega) \in I \times \Omega$, and the feasibility constraint (7). An incentive-feasible allocation $(\hat{c}, \hat{N})$ *Pareto dominates* the incentive-feasible allocation (c, N) if $U_i(\hat{c}_i) \geq U_i(c_i)$ for all $i \in I$, with at least one strict inequality for some $i \in I$. An allocation is *incentive-constrained Pareto optimal* if it is incentive-feasible and not Pareto dominated by any other incentive-feasible allocation. In our model, we have:

Proposition 1 *Any equilibrium allocation is incentive-constrained Pareto optimal.*

As in Prescott and Townsend (1984), while incentive compatibility constrains consumption, consumption sets remain convex, and equilibrium is constrained Pareto optimal. Thus, the proof is similar to its perfect market counterpart: if an equilibrium allocation was Pareto dominated by another incentive feasible allocation, the latter must lie outside the agents' budget set. Adding up across agents leads to a contradiction. Intuitively, the reason why optimality obtains in spite of incentive constraints is because prices do not show up in the incentive compatibility condition, so that there are no "contractual externalities".

3.3.2 Existence and Uniqueness

To prove existence of equilibrium, we follow the standard approach of Negishi (1960). Namely, we consider the problem of a planner who assigns Pareto weights $\alpha_i \geq 0$ to each agent $i \in I$, with $\sum_{i \in I} \alpha_i = 1$, and then chooses incentive feasible allocations to maximize the social welfare function, $\sum_{i \in I} \alpha_i U_i(c_i)$. We establish the existence of Pareto weights such that, given agents' initial endowment, the social optimum can be implemented in a competitive equilibrium without making any wealth transfers between agents.

Proposition 2 *There exists an equilibrium.*

The proof follows arguments found in Negishi (1960), Magill (1981), and Mas-Colell and Zame (1991) with a few differences. First, our planner is now subject to incentive compatibility constraints. Second, utility functions may be unbounded below and so not continuous at $c = 0$. Third, technical difficulties arise because the commodity space is infinite dimensional. In particular, the set $\mathcal{M}_+$ of positive measures has an empty interior when viewed as a subset of the space of signed measures endowed with the total-variation norm. This creates difficulty in applying separation theorems: in the language of Mas-Colell and Zame (1991), "preferred set may not be supportable by prices". In the context of our model, we solve this difficulty by deriving first-order necessary and sufficient conditions for the Planner's problem, and using the associated Lagrange multipliers to construct equilibrium prices.

We can show uniqueness in a particular case of interest:

Proposition 3 *Suppose that there are two types of agents, $I = \{1, 2\}$, with CRRA utility, with respective RRA coefficients (γ_1, γ_2) such that $0 \leq \gamma_1 \leq \gamma_2 \leq 1$ and $\gamma_2 > 0$. Then the equilibrium consumption allocation is uniquely determined. The prices of Arrow securities and the price of trees, $\bar{N}$-almost everywhere, are all uniquely determined up to a positive multiplicative constant.*

In general, the asset allocation is not uniquely determined. As will be clear below, this arises for example when none of the incentive constraints bind. In that case the allocation is not uniquely determined because it is equivalent to hold tree j or a portfolio of Arrow securities with the same cash-flows as j.

As is standard, only relative prices are pinned down, hence price levels are only determined up to a positive multiplicative constant.

Finally, asset prices are only uniquely determined $\bar{N}$-almost everywhere. In particular, the prices of assets in zero supply are not uniquely determined. This is intuitive: given the short-sale constraint, the only equilibrium requirement for an asset in zero supply is that the price is large enough so that no agent want to hold it. As a result equilibrium only imposes a lower bound on the price of trees in zero supply.

3.3.3 Arbitrage

Lemma 4 *The following no-arbitrage relationships must hold:*

- *Trees and Arrow securities have strictly positive prices: $p_j > 0$ for all $j \in [0, 1]$ and $q(\omega) > 0$ for all $\omega \in \Omega$;*

- *The prices of trees in positive supply are lower than or equal to the prices of the portfolios of Arrow securities with the same payoff. That is, $\bar{N}$-almost everywhere, $p_j \leq \sum_{\omega \in \Omega} q(\omega) d_j(\omega)$.*

Absence of arbitrage requires that Arrow securities and tree prices be positive, for standard reasons. It also implies that the prices of trees cannot be above those of portfolios of Arrow securities with the same cash flows. If it were, this would open an arbitrage opportunity, which agents could exploit by selling trees in positive supply and buying portfolios of Arrow securities. Such arbitrage would be possible because i) trees are in positive net supply and so selling these trees is feasible for at least one agent ii) buying Arrow securities does not tighten incentive compatibility constraints. In contrast, if the prices of trees are below those of corresponding portfolios of Arrow securities, arbitrage would require selling those securities. This would tighten incentive compatibility constraints, however. Thus, as shown below, it can be the case in equilibrium, when incentive compatibility constraints are binding, that the price of trees is strictly lower than that of corresponding portfolios of Arrow securities. This is a form of limit to arbitrage.

3.3.4 Implementability

We first study circumstances under which the incentive compatibility constraints do not impact equilibrium outcomes. Formally, define a $\delta = 0$ equilibrium to be an allocation and price system (c^0, N^0, p^0, q^0) when $\delta = 0$, i.e., when agents have no ability to divert. Fix some $\delta > 0$. Then, the $\delta = 0$-equilibrium is said to be $\delta > 0$-*implementable* if there exists some $\delta > 0$-equilibrium, $(c^\delta, N^\delta, q^\delta, p^\delta)$, such that $c^0 = c^\delta$. The next lemma states an intuitive necessary and sufficient condition for implementability:

Lemma 5 *Fix some $\delta > 0$. Then, a $\delta = 0$-equilibrium, (c^0, N^0, p^0, q^0), is $\delta > 0$-implementable if and only if there exists some $N^\delta = (N_i^\delta)_{i \in I}$ such that :*

$$\sum_{i \in I} N_i^\delta = \bar{N} \tag{9}$$

$$c_i^0(\omega) \geq \delta \int d_j(\omega)\, dN_{ij}^\delta \quad \forall (i, \omega) \in I \times \Omega. \tag{10}$$

This yields the following proposition:

Proposition 6 *Fix some $\delta > 0$. A $\delta = 0$-equilibrium (c^0, n^0, p^0, q^0) is $\delta > 0$-implementable if one of the following conditions is satisfied:*

- *There exists $\{N_i\}_{i \in I} \in \mathcal{M}_+^{|I|}$ such that $\sum_{i \in I} N_i = \bar{N}$ and $\int d_j(\omega)\, dN_{ij} = c_i^0(\omega)$ $\forall (i, \omega) \in I \times \Omega$.*
- *Agents have Constant Relative Risk Aversion (CRRA) with identical coefficient.*

The first bullet point of the proposition states that the incentive compatibility constraint is satisfied if two conditions are satisfied. First agents can replicate their zero-equilibrium consumption with *positive* holdings of trees. Second, these agents holding are *feasible*, i.e., they add up to the aggregate. This means that they do not need to make any financial promise, i.e., promise to deliver consumption out of the payoff of their equilibrium holdings of trees. Clearly, if agents do not need to make any financial promise, divertibility is not an issue. Keep in mind, however, that the feasibility condition is crucial. In particular, in Section 4, we will provide an example in which the asset structure is very rich: it includes assets with payoffs proportional to agents' consumption profiles in the zero-equilibrium. Yet, the zero-equilibrium is not implementable with $\delta > 0$. The reason is that, in equilibrium, agents must hold the entire asset supply. In particular they will have to hold portfolios whose payoffs differ from their desired consumption profiles. As a result, they will have to issue liabilities and run into incentive problems.

The second bullet point is an example of the first: if agents have CRRA utilities with identical risk aversion, then they all consume a constant share of the aggregate endowment. Clearly, they can attain that consumption plan by holding a portfolio of trees, namely a constant share in the market portfolio.

3.4 Optimality conditions

Since agents have concave objectives and are subject to finite-dimensional affine constraints, the interior point condition for the positive cone associated with the constraint set is immediately satisfied, so one can apply the Lagrange multiplier Theorems shown in Section 8.3 and 8.3 of Luenberger (1969) (see Proposition 18 in the appendix for details). Let λ_i denote the Lagrange multiplier of the intertemporal budget constraint (6) and $\mu_i(\omega)$ the Lagrange multiplier of the incentive compatibility constraint (5). The first-order condition with respect to $c_i(\omega)$ is:

$$\pi(\omega)u_i'\left[c_i(\omega)\right] + \mu_i(\omega) = \lambda_i q(\omega). \tag{11}$$

In particular, it can be shown that there exists multipliers that make this condition hold at equality even when $c_i(\omega) = 0$. The first-order condition with respect to N_i can be written

$$p_j \geq \sum_{\omega\in\Omega} q(\omega)d_j(\omega) - \delta \sum_{\omega\in\Omega} \frac{\mu_i(\omega)}{\lambda_i} d_j(\omega) \tag{12}$$

with an equality N_i–almost everywhere, that is, for almost all trees held by agent i.

3.4.1 Asset pricing

The pricing of risk and incentives. The pricing kernel, pricing the Arrow securities is

$$M(\omega) \equiv \frac{q(\omega)}{\pi(\omega)}.$$

The first order condition with respect to consumption, (11), shows that if the incentive compatibility conditions were slack, the marginal rate of substitution between consumptions in different states would be equal across all agents, as in the standard, perfect and complete markets, model. When incentive compatibility conditions bind, in contrast, marginal rates of substitution differ across agents, reflecting the multipliers of the incentive constraints. This reflects imperfect risk-sharing in markets that are endogenously incomplete due to incentive constraints, as in Alvarez and Jermann (2000). Thus the Arrow securities pricing kernel arising in our model differs from its complete or exogenously incomplete markets counterpart because in general, there is no agent whose marginal utility is equal to $M(\omega)$ *in all states.* Instead, $M(\omega)$ corresponds to the marginal utility of an unconstrained agent, whose type varies from state to state.

Denote

$$A_i(\omega) \equiv \frac{\mu_i(\omega)}{\lambda_i \pi(\omega)},$$

which can be interpreted as the shadow cost of the incentive compatibility constraint of agent i in state ω. With these notations, (12) rewrites as:

$$p_j \geq \mathbb{E}\left[M(\omega)d_j(\omega)\right] - \delta\,\mathbb{E}\left[A_i(\omega)d_j(\omega)\right], \tag{13}$$

with an equality for almost all trees held by agent i. Equation (13) shows that the price of an asset held by i is the difference between two terms.

The first term is $\mathbb{E}\left[M(\omega)d_j(\omega)\right]$, the present value of the dividends evaluated with the pricing kernel M. It reflects the pricing of risk embedded in the the prices of the Arrow securities.

The second term, $\delta\,\mathbb{E}\left[A_i(\omega)d_j(\omega)\right]$, is new to our setting. It reflects the pricing of incentives, as it is equal to the shadow cost incurred by agents of type i when they hold one marginal unit of asset j and their incentive constraints becomes tighter. It is the expected product of the shadow cost of the incentive constraint, $A_i(\omega)$, and of the divertible dividend flow, $\delta\, d_j(\omega)$.

Excess return decomposition. The pricing formula (13) also leads to a natural decomposition of excess return. Define the risky return on asset j as $R_j(\omega) \equiv d_j(\omega)/p_j$ and let the risk-free return be $R_f \equiv 1/\mathbb{E}\left[M(\omega)\right]$. Then, standard manipulations of the first order condition (12) shows that for almost all assets held by agents of type i:

$$\mathbb{E}\left[R_j(\omega)\right] - R_f = -R_f \operatorname{cov}\left[M(\omega), R_j(\omega)\right] + R_f\,\mathbb{E}\left[A_i(\omega)\delta R_j(\omega)\right] \tag{14}$$

The first term on the right-hand-side of (14) can be interpreted as a risk premium. It is positive if the return on asset j, $R_j(\omega)$, is large for states in which the pricing kernel, $M(\omega)$, is low. It is similar to the standard risk-premium obtained in frictionless markets (see, e.g., Huang and Litzenberger (1988) equation 6.2.8) but, unlike in the frictionless CCAPM, the pricing kernel $M(\omega)$ mirror neither aggregate nor individual consumption.

The second term on the right-hand-side of (14) can be interpreted as a divertibility premium. It is positive if divertible income, $\delta R_j(\omega)$, large when the incentive compatibility condition of the agent holding the asset binds.

Limits to arbitrage. Lemma 4 stated that, by arbitrage, the price of a tree could not be larger than the price of a corresponding portfolio of Arrow securities delivering the same cash flows. Equation (13) reveals further that, if the incentive compatibility constraint of the asset holder binds in at least one state, and if the dividend is strictly positive in that state, then the price of the tree is *strictly* smaller than that of the corresponding portfolio of Arrow securities. One may argue that this constitutes an arbitrage opportunity. However, agents of type i cannot trade on it without tightening their incentive constraint. Thus, the wedge between $\mathbb{E}\left[M(\omega)d_j(\omega)\right]$ and the price, p_j, can be interpreted as a divertibility discount, arising because of limits to arbitrage.

One natural empirical counterpart of the discount is the "basis" often observed between real assets and replicating portfolios involving derivatives. Examples include the basis between a Treasury and a portfolio of corporate bond and CDS, or the basis between a future and the underlying stock or commodity. While there are many potential empirical drivers of the basis, our model makes a strong prediction. Namely, incentive constraints always open the basis in one direction, with real assets being *underpriced* relative to the corresponding replicating portfolio of derivatives.

Divertibility discount vs. collateral premium. While our model points to a "divertibility discount," our results can also be interpreted in terms of premium, but relative to a different benchmark. To see this, consider again the trees held by some agent i. Take the first-order condition (11) with respect to $c_i(\omega)$, multiply by the dividend $d_j(\omega)$ and sum across states to obtain:

$$\mathbb{E}\left[M(\omega)d_j(\omega)\right] = \mathbb{E}\left[\frac{u_i'\left[c_i(\omega)\right]}{\lambda_i} d_j(\omega)\right] + \mathbb{E}\left[A_i(\omega)d_j(\omega)\right]. \tag{15}$$

Substituting (15) into (13) asset j is

$$p_j = \mathbb{E}\left[\frac{u_i'\left[c_i(\omega)\right]}{\lambda_i} d_j(\omega)\right] + \mathbb{E}\left[A_i(\omega)d_j(\omega)\right] - \delta\, \mathbb{E}\left[A_i(\omega)d_j(\omega)\right]. \tag{16}$$

This price equation is similar to equation (5) in Fostel and Geanakoplos (2008) or that in Lemma 5.1 in Alvarez and Jermann (2000). The first term on the right-hand side of (16) is similar to what Fostel and Geanakoplos (2008) call "payoff value": it is the expected value of asset's cash flows, evaluated at the marginal utility of the agent holding the asset (it reflects both the expectation of the dividend, and its covariance with the agent's marginal utility, usually interpreted in terms of risk premium). The second term on the right-hand side of (16) is similar to the collateral premium in Fostel and Geanakoplos (2008) (see Lemma 1, page 1230). The third term is the divertibility discount, which is specific to our model, and does not arise in Fostel and Geanakoplos (2008).

3.4.2 Segmentation

Let

$$v_{ij} = \mathbb{E}\left[M(\omega)d_j(\omega)\right] - \delta\,\mathbb{E}\left[A_i(\omega)d_j(\omega)\right] \tag{17}$$

denote the valuation of agent i for asset j. From the first-order condition (12), one sees that $v_{ij} = p_j$ for almost all asset held by agents of type i, and otherwise $v_{ij} \leq p_j$. Therefore, the agents who hold the asset are those who value it the most, because they have the lowest shadow incentive-cost of holding the assets.

In the general model, we have found it difficult to provide a sharp characterization of the equilibrium asset allocation. But this can be done in the context of particular examples, such as the one developed in Section 4 below. In this example, different assets are held, in equilibrium, by different agents. This equilibrium outcome resembles the one exogenously assumed in the segmented market literature, in particular recent work on "intermediary asset pricing" (see for example Edmond and Weill (2012) or He and Krishnamurthy (2013)). However, the pricing formula differs from that in exogenously segmented market. Namely, in our endogenously segmented markets, assets are not priced by the marginal utility of the asset holders and they include a divertibility discount. Also, the extent of segmentation is determined in equilibrium and so will not be invariant to changes in the economic environment.

4 Two-by-Two

To obtain more explicit equilibrium properties, in particular to characterize the asset allocation more precisely, we hereafter focus on the simple "two-by-two" case, in which there are two types of agents, two states, and an arbitrary distribution of assets. We further assume that both types of agents, $i \in \{1, 2\}$, have CRRA utility with respective coefficient of relative risk aversion $0 \leq \gamma_1 < \gamma_2 \leq 1$. That is, agent $i = 1$ is more risk-tolerant, while agent $i = 2$ is more risk-averse. As shown in Proposition 3, this implies that the equilibrium consumption allocation is uniquely determined, and the equilibrium prices are uniquely determined up to a multiplicative constant. As shown in Proposition 6, the restriction $\gamma_1 \neq \gamma_2$ is necessary for incentive compatibility to matter in equilibrium.

We normalize the dividend of each asset to one, i.e., $\mathbb{E}\left[d_j(\omega)\right] = 1$.[6] Given that there are only two states, all trees must lie in the convex hull of two extreme securities: one that only pays off in state ω_1, and one that only pays off in state ω_2. Therefore, one can order the trees so that, for any $j \in [0, 1]$,

$$d_j(\omega) = \frac{j}{\pi(\omega_1)} \mathbb{I}_{\{\omega=\omega_1\}} + \frac{1-j}{\pi(\omega_2)} \mathbb{I}_{\{\omega=\omega_1\}}.$$

We label the states by $\{\omega_1, \omega_2\}$ such that the aggregate endowment, denoted by $y(\omega) = \int d_j(\omega)\, d\bar{N}_j$, is strictly larger in state ω_2 than in state ω_1:

$$y(\omega_2) = \frac{1}{\pi(\omega_2)} \int (1-j)\, d\bar{N}_j > y(\omega_1) = \frac{1}{\pi(\omega_1)} \int j\, d\bar{N}_j.$$

In other words, ω_1 is the "bad state" while ω_2 is the "good state." The tree $j = \pi(\omega_1)$ is risk free, and so its aggregate endowment beta, $\mathrm{cov}\left[d_j(\omega), y(\omega)\right]/V\left[y(\omega)\right]$ is zero. Trees with $j < \pi(\omega_1)$ have lower dividend in state ω_1 than in state ω_2, and so have positive aggregate endowment beta. The smaller is j, the the more positive is the beta. Vice versa, trees with $j > \pi(\omega_1)$ have negative aggregate endowment beta. The larger is j, the more negative is the beta.

[6] This is without loss of generality. This merely amounts to divide the dividend in all states by the expected dividend, and simultaneously scaling the asset supply up by the same constant.

4.1 Incentive feasible consumption allocations

We start by studying the set of *incentive feasible* consumption allocations, that is, consumption allocations c such that (c, N) is incentive feasible for some tree allocation N. This simplifies the analysis by reducing the number of choice variables: it allows to work directly with consumption allocations, without having to explicitly describe the underlying asset allocation that makes it incentive compatible. In particular, it allows to analyze incentive-feasibility and equilibrium in an Edgeworth box. Our first main result is:

Proposition 7 *Consider a feasible consumption allocation such that $c_1(\omega_1)/y(\omega_1) < c_1(\omega_2)/y(\omega_2)$. Then c is incentive feasible if and only if there exists $k \in [0, 1]$ and $(\Delta N_1, \Delta N_2) \geq 0$, $\Delta N_1 + \Delta N_2 = \bar{N}_k - \bar{N}_{k-}$, such that:*

$$c_1(\omega_1) \geq \delta \int_{j\in[0,k)} d_j(\omega_1) d\bar{N}_j + \delta d_k(\omega_1)\Delta N_1 \tag{18}$$

$$c_2(\omega_2) \geq \delta \int_{j\in(k,1]} d_j(\omega_2) d\bar{N}_j + \delta d_k(\omega_2)\Delta N_2. \tag{19}$$

The proposition focuses on the case in which the consumption share of agent 1 is lower in the bad state than in the good state. This is natural since agent 1 is more risk tolerant than agent 2, and, in the first best, its consumption share is indeed lower in the bad than in the good state. The result stated in the proposition follows from two observations.

The first observation is that, since his consumption share is smaller in ω_1 than in ω_2, agent $i = 1$ tends to have incentive problems in state ω_1. To understand why, imagine that agent $i = 1$ purchases a fraction of the market portfolio equal to her average consumption share across states. In order to implement his consumption plan $c_1(\omega)$ while holding this portfolio, agent $i = 1$ has to sell Arrow securities that pay off in state ω_1, and purchase Arrow securities that payoff in state ω_2. Hence, agent $i = 1$ only has a liability in state ω_1, and so only has incentive to divert in that state. Vice versa, agent $i = 2$ tends to have incentives to divert in state ω_2.

The second observation is that, in this context, in order to mitigate these incentive problems, it is best to allocate agent $i = 1$ a portfolio of trees with low payoff in state ω_1. This minimizes agent $i = 1$ incentive to divert. Vice versa, it is best to allocate agent $i = 2$ a portfolio of trees with low payoff in state ω_2. Since we have ordered trees so that the payoff in state ω_1 is strictly increasing in j, feasibility then implies that agent $i = 1$ should receive all trees $j < k$, and agent $i = 2$ all trees $j > k$, for some threshold k. The proposition states, then, that a consumption allocation is incentive feasible if and only if the incentive compatibility constraints

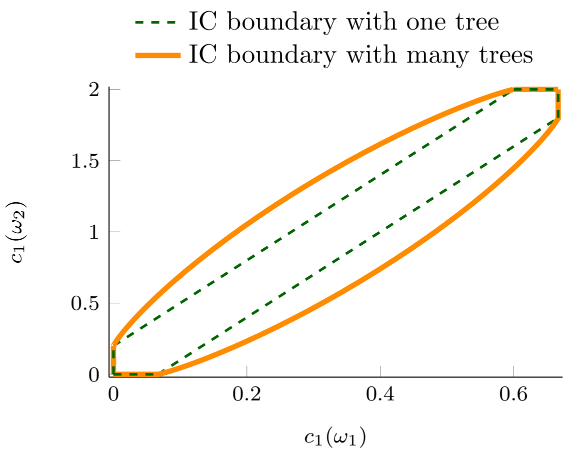

Figure 1: The set of incentive feasible consumption allocations. In the many-trees case, tree supplies are distributed according to a beta distribution with parameters $a = b = 15$. In the one-tree case, there is just one tree equal to the market portfolio of the many-trees case. The probability of the high state is $\pi(\omega_2) = 0.25$. The divertibility parameter is $\delta = 0.9$.

hold for such a portfolio.

The right-hand sides of (18) and (19) define a boundary below which any consumption allocation above the diagonal of the Edgeworth box is incentive feasible, and above which it is not. The case of allocations below the diagonal is not explicitly addressed in the proposition because it is just symmetric. Figure 1 illustrates. The consumption of agent $i = 1$ in state ω_1 is on the x-axis, and his consumption in state ω_2 is on the y-axis. The dashed line is the boundary of the incentive-feasible set when there is just one tree in strictly positive supply.[7] The solid line is the boundary when there are many trees.[8] As expected, the incentive-feasible set is convex. It is smaller with one tree than with many trees. Indeed, with many trees, one can replicate one-tree allocations by allocating agents shares in the market portfolio. Also, one sees in the figure that any sufficient small consumption allocation $(c_1(\omega_1), c_1(\omega_2))$ is incentive feasible. Indeed, as long as $\delta < 1$, such a consumption allocation can be made incentive feasible by allocating most of the trees to agent $i = 2$.

[7] In that case, the distribution $\bar{N}$ has just one atom. If we normalize this atom to one for simplicity, then in the Edgeworth box the boundary is the curve parameterized by $\Delta N_1 \in [0, 1]$, with cartesian coordinates $c_1(\omega_1) = \delta d(\omega_1)\Delta N_1$ and $c_1(\omega_2) = y(\omega_2) - c_2(\omega_2) = d(\omega_2)\,[1 - \delta + \delta\Delta N_1]$

[8] In that case we assume no atom, so the boundary is the curve parameterized by $k \in [0, 1]$, with cartesian coordinates $c_1(\omega_1) = \delta \int_0^k d_j(\omega_1)\, d\bar{N}_j$ and $c_1(\omega_2) = y(\omega_2) - c_2(\omega_2) = \int_0^1 d_j(\omega_2)\, d\bar{N}_j - \delta \int_k^1 d_j(\omega_2) d\bar{N}_j$.

A useful property for what follows is that, for any incentive-feasible consumption allocation on the boundary, the distribution of assets is uniquely determined.

Proposition 8 *Suppose that (18) and (19) holds with equality for some consumption allocation c, some $k \in [0,1]$ and some $(\Delta N_1, \Delta N_2) \geq 0$ such that $\Delta N_1 + \Delta N_2 = N_k - N_{k-}$. Then (c, N) is incentive feasible if and only if $N_1 = \Delta N_1 \mathbb{I}_{\{j=k\}} + \bar{N}\, \mathbb{I}_{\{j<k\}}$ and $N_2 = \Delta N_2 \mathbb{I}_{\{j=k\}} + \bar{N}\, \mathbb{I}_{\{j>k\}}$.*

Consider the simple case in which there are no atoms in the distribution of assets. Then $\Delta N_1 = \Delta N_2 = 0$ and the proposition states that there exists a k such that agent 1 holds assets $j \leq k$, while agent 2 holds assets $j > k$. When the distribution is not atomless, things are a bit more complicated when there is an atom at k. This is trivially the case, for example, if there was only one asset. Then the mass of assets from $j = 0$ to $j = N_{k-}$ (just below k) is strictly lower than the mass of assets from $j = 0$ to $j = k$. That is $N_k - N_{k-} > 0$. In that case both agents hold some of asset k. Out of the total mass of asset k, $N_k - N_{k-}$, agent 1 holds a mass ΔN_1 while agent 2 holds a mass ΔN_2.

4.2 Equilibrium allocations

In order to characterize equilibrium allocations, we rely on their efficiency properties. Let (c, N) denote the equilibrium allocation. As shown in Proposition 1, (c, N) is constrained Pareto efficient. Combining the proof of Proposition 2 and Proposition 7, we know that c solves an *incentive-constrained* Planner's problem. That is, there exists weights $(\alpha_1, \alpha_2) \in (0,1)^2$, $\alpha_1 + \alpha_2 = 1$, such that c maximizes $\sum_{i \in I} \alpha_i U_i(c_i)$ with respect to feasible allocations satisfying the incentive compatibility conditions (18) and (19). Let $c^\star$ denote the solution of the corresponding *unconstrained* Planner's problem. That is, $c^\star$ maximizes the same welfare function, with the same weights (α_1, α_2), with respect to feasible allocations, but without imposing the incentive compatibility conditions.

Lemma 9 *If $(\alpha_1, \alpha_2) > 0$, then the solutions of the unconstrained and incentive-constrained Planner's problems both lie strictly above the diagonal of the Edgeworth box. That is $c_1^\star(\omega_1)/y(\omega_1) < c_1^\star(\omega_2)/y(\omega_2)$ and $c_1(\omega_1)/y(\omega_1) < c_1(\omega_2)/y(\omega_2)$.*

The lemma states that the risk-tolerant agent, $i = 1$, receives a lower share of aggregate consumption in the low state than in the high state (as in the first best). Since consumption shares add up to one across agents, it follows that the risk-averse agent, $i = 2$, enjoys a higher share of aggregate consumption in the low than in the high state. Intuitively, a consumption allocation which delivers a constant consumption share in both states to both agents is always strictly incentive feasible: it can be implemented by giving each agent a share in the market portfolio equal to that consumption share. But the risk-tolerant cares relatively less about the low state, ω_1, and relatively more about the high state, ω_2. Hence, social welfare increases strictly if the risk-tolerant agent, $i = 1$ insures the more risk-averse agent by letting $i = 2$ have a larger share of aggregate consumption in the bad state.

One implication of the proposition is that the planner always find it optimal to pick consumption allocations above the diagonal of the Edgeworth box. Therefore, the relevant incentive constraint is the upper boundary of the incentive compatible set in Figure 1. Together with Proposition 8, this implies:

Corollary 10 *If $c \neq c^\star$, then both (18) and (19) must bind for some $k \in [0, 1]$ and $(\Delta N_1, \Delta N_2) \geq 0$ such that $\Delta N_1 + \Delta N_2 = \bar{N}_k - \bar{N}_{k-}$. The incentive compatibility constraint of agent $i = 1$ binds in state ω_1 and agent $i = 1$ holds all assets $j < k$. Likewise, the incentive compatibility constraint of agent $i = 2$ binds in state ω_2 and agent $i = 2$ holds all assets $j > k$.*

The corollary is illustrated in Figure 2. In the figure, the "incentive-constrained Pareto set" and the "unconstrained Pareto set" are, respectively, the set of consumption allocations obtained by solving the incentive-constrained and the constrained Planner's problem for all possible weights $(\alpha_1, \alpha_2) \in [0, 1]^2$, $\alpha_1 + \alpha_2 = 1$. The incentive-constrained Pareto set coincides with the unconstrained Pareto set when the later lies below the IC boundary. Otherwise, the incentive-constrained Pareto set coincides with the IC boundary. As α_1/α_2 increases, then the constrained Pareto efficient allocation move monotonically to the northeast of the Edgeworth box.

The figure reveals that incentive compatibility does not matter for extreme values of α_1/α_2. For example, when α_1/α_2 is close to infinity, unconstrained Pareto efficiency requires that agent $i = 1$ receives almost all of the output. When $\delta < 1$, such an allocation is incentive compatible if agent $i = 1$ holds all the trees. In equilibrium, agent $i = 1$ purchases all the assets and issues a liability to agent $i = 2$ with payoff $c_2(\omega)$, equal to the consumption plan of agent $i = 2$. Because agent $i = 2$ does not consume much, the liability is smaller than agent $i = 1$'s non-divertible income, so $i = 1$ does not have incentives to default.

In the example of the figure, incentive compatibility matters for intermediate values of α_1/α_2. This arises because, in the unconstrained Pareto set, the consumption plans of both agents differ significantly from the payoff of the market portfolio. In an equilibrium, the implementation of such consumption plans requires that both agents issue significant liabilities to each others, giving rise to incentive problems.

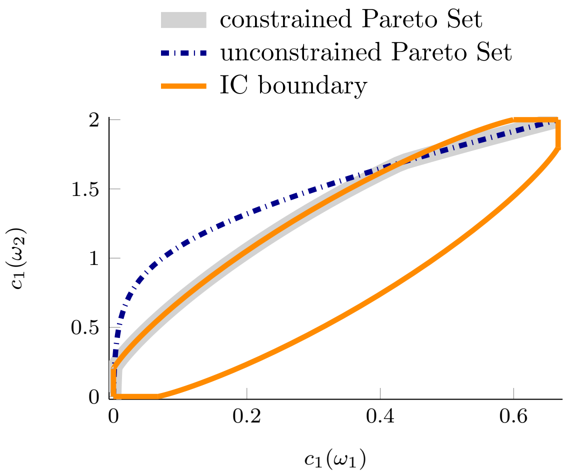

Figure 2: The set of incentive-constrained. The RRA of agent $i = 1$ is $\gamma_1 = 0.3$ and that of agent $i = 2$ is $\gamma_2 = 1$. The other parameters are the same as in Figure 1.

Finally, the characterization so far has been done in terms of the endogenous Pareto weigths (α_1, α_2) and not in terms of the primitive exogenous initial endowments $(\bar{n}_1, \bar{n}_2)$. In the two-by-two case, a corollary of our existence proof is:

Corollary 11 *The ratio of endogenous Pareto weights,* α_1/α_2*, is strictly increasing in the ratio of initial endowment* $\bar{n}_1/\bar{n}_2$*.*

This implies in particular that, as $\bar{n}_1/\bar{n}_2$ increases, then the equilibrium allocation moves monotonically to the northeast of the Edgeworth box along the incentive-constrained Pareto set. It also implies that incentive problems only arise for intermediate values of $\bar{n}_1/\bar{n}_2$, that is, when the distribution of wealth is not too concentrated.

4.3 Asset pricing

4.3.1 Cross sectional divertibility discounts

Equation (12) shows that there is a wedge between the price of trees and the price of the portfolios of Arrow securities with the same cash flows. This wedge is equal to the shadow cost of tightening the IC constraint for agents holding the tree. In the two-by-two case, the first order condition with respect to asset holdings, (12) simplifies to

$$\sum_{\omega \in \Omega} q(\omega) d_j(\omega) - p_j = \delta \frac{\mu_1(\omega_1)}{\lambda_1} d_j(\omega_1),$$

for all tree $j \leq k$, which are held by agent $i = 1$.[9]

In what follows we will state cross-sectional implications and conduct comparative statics for the wedge. Since only relative prices are pinned down, we express the divertibility discount in relative price, and choose as normalizing factor (or numeraire) the price of the riskless bond $1/R_f$. Now, the risk free rate is the inverse of the sum of state prices. State prices are pinned down by the first order condition with respect to consumption of the unconstrained agent $q(\omega_i) = \frac{1}{\lambda_{-i}} \pi(\omega_i) u'_{-i} \left[c_{-i}(\omega_i)\right]$. It follows that, in our simple two-by-two case, the price of the riskless bond is

$$\frac{1}{R_f} = \sum_{\omega \in \Omega} q(\omega) = \frac{1}{\lambda_2} \pi(\omega_1) u'_2 \left[c_2(\omega_1)\right] + \frac{1}{\lambda_1} \pi(\omega_2) u'_1 \left[c_1(\omega_2)\right].$$

We now focus on the divertibility discount normalized by the risk free rate. For tree $j < k$ this is

$$\Delta_j \equiv \frac{\sum_\omega q(\omega) d_j(\omega) - p_j}{R_f} \tag{20}$$

Thus

$$\Delta_j = \frac{\lambda_2 \mu_1(\omega_1)}{\pi(\omega_1) \lambda_1 u'_2(c_2(\omega_1)) + \pi(\omega_2) \lambda_2 u'_1(c_1(\omega_2))} \, \delta d_j(\omega_1). \tag{21}$$

The right-hand side of equation (21) is the product of two terms. The first term is constant across all assets held by agent 1, and measures, intuitively, the tightness of the incentive constraint of agent 1. The second term is equal to the divertible cash flow of the asset in the state in which the agent holding it is constrained. Among assets held by the risk-tolerant agent, $i = 1$, this term, and correspondingly the divertibility discount, is

[9] If there is an atom in the distribution of assets at k, then both agents' types hold asset k. Correspondingly $\frac{\mu_1(\omega_1)}{\lambda_1} d_k(\omega_1) = \frac{\mu_2(\omega_2)}{\lambda_2} d_k(\omega_2)$.

higher for assets with a relatively large payoff in the bad state and a relatively low payoff in the high state, that is, assets with a lower aggregate endowment beta. The intuition is that the risk tolerant agent sells insurance against the bad state to the risk-averse agent. However, the incentive compatibility constraint limits the amount of insurance she can sell. Since the consumption of the risk-tolerant agent is low in the bad state, diverting cash flows of trees she holds is tempting. It implies that the shadow cost of holding a tree is higher for trees paying relatively more in the bad state, i.e., for trees with a lower aggregate endowment beta. Remember however that the risk-tolerant agent holds trees with a high betas. Therefore, among trees with a high aggregate endowment beta, trees with a moderately high beta have a larger divertibility discount than trees with a very high beta.

Consider now trees $j > k$ held by agent 2. Following the same reasoning as before, the divertibility discount equals

$$\Delta_j \equiv \frac{\sum_\omega q(\omega) d_j(\omega) - p_j}{\sum_\omega q(\omega)} = \frac{\lambda_1 \mu_2(\omega_2)}{\pi(\omega_1)\lambda_1 u_2'(c_2(\omega_1)) + \pi(\omega_2)\lambda_2 u_1'(c_1(\omega_2))} \, \delta d_j(\omega_2). \tag{22}$$

Equation (22) implies that, among assets held by the risk averse agent, $i = 2$, the divertibility discount is higher for assets with a relatively large payoff in the good state and a relatively low payoff in the bad state, that is, with a higher aggregate endowment beta. The intuition is symmetric to the one above. The risk-averse agent would like to sell consumption to the risk tolerant agent in the good state, but it is tempting for the risk averse agent to divert the cash flows of the trees he holds in the good state. Thus, the shadow cost of holding a tree is higher for tree with a relatively high payoff in the good state, that is, for trees with a higher aggregate endowment beta. The risk averse agent holds trees with a low aggregate endowment beta. Therefore, among trees with a low beta, those with a moderately low beta have a lager divertibility discount than trees with a very low beta. Putting things together, we conclude that:

Lemma 12 *Suppose the distribution of tree supplies is strictly increasing. Then, the divertibility discount is an inverse U-shape function of the aggregate endowment beta of the tree.*

The restriction that the distribution is strictly increasing means that all trees are in positive supply and so that their prices are uniquely determined. This intuitively means that, after adjusting for risk, trees with either a low or a large aggregate endowment beta will tend to have a high price, and a low return. This is illustrated in the next figure. The figure shows the security market line (SML) in our environment, which we derive explicitly in Proposition 24 in Supplementary Appendix B.7.2. Since assets are held by agents who value them most, the SML is the minimum between the SML obtained from agent $i = 1$'s valuation, and that derived from agent $i = 2$'s valuation. The kink in the figure occurs at asset k, for which ownership switches from agent 1 to agent 2. The figure illustrates that, because the divertibility discount is inverse-U shaped in β, the SML is flatter at the top, in line with Black (1972), and recent evidence in Frazzini and Pedersen (2014) and Hong and Sraer (2016).

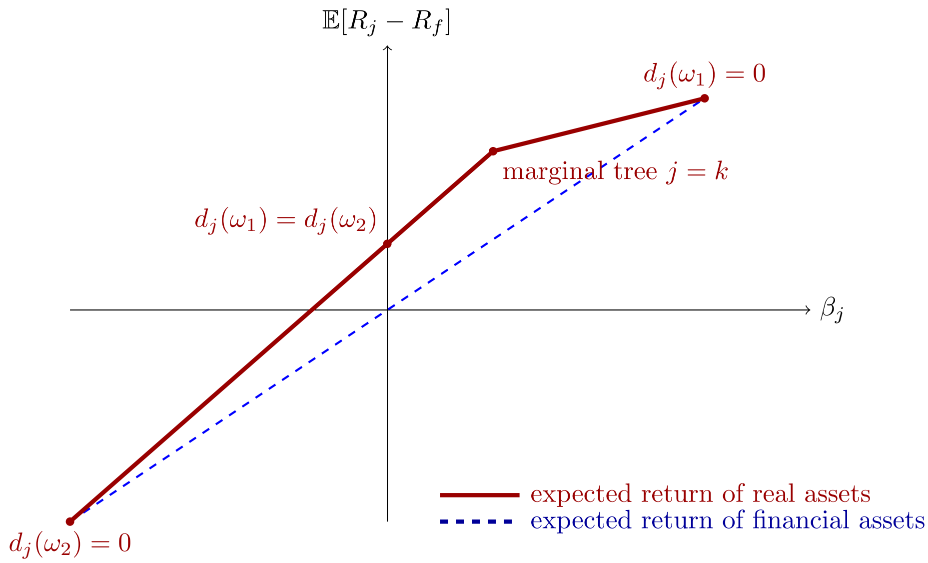

Figure 3: Modified security market line.

4.3.2 Comovements in divertibility discounts

What is the effect of a tree's δ on its own divertibility discount and on the divertibility discount of the other trees? Fix a tree $\ell < k$ and consider a small increase in δ for tree ℓ and possibly nearby trees. Formally we assume $\delta_j = \delta + \varepsilon \phi_j$ for some continuous function ϕ_j strictly positive near ℓ, and zero everywhere else.[10] This allows us to establish:

Lemma 13 *Assume that the cumulative distribution of trees is continuous and strictly increasing, that $c \neq c^\star$, and that $k \in (0, 1)$. Then, an increase in ε shrinks the set of trees held by agent 1: $k(\varepsilon') < k(\varepsilon)$ for small $\varepsilon' > \varepsilon$.*

[10] All of our results extend to this case. In fact, our proofs in the appendix cover the case of δ which are continuously varying across agents and asset types.

When agent 1 becomes slightly worse at pledging a tree he already holds, the shadow value of his incentive-compatibility constraint increases, which makes it more costly for agent 1 to hold other trees. Thus, in equilibrium, the set of trees $[0, k)$ held by agent 1 shrinks. What is the effect on divertibility discounts? Clearly, the divertibility discount of tree ℓ increases relative to other trees held by agent 1

$$\frac{\Delta_\ell}{\Delta_j}(\varepsilon') > \frac{\Delta_\ell}{\Delta_j}(\varepsilon)$$

for $\varepsilon' > \varepsilon$ and for all $j < k$ such that $\phi_j = 0$. What is the effect for other trees? For two trees held by agent 1 ($j, j' < k$ such that $\phi_j = \phi_{j'} = 0$), equation (21) implies that their divertibility discounts change at the same rate:

$$\frac{\Delta_j}{\Delta_{j'}}(\varepsilon') = \frac{\Delta_j}{\Delta_{j'}}(\varepsilon),$$

for $\varepsilon' > \varepsilon$. Now, consider two trees $j < k$ held by agent 1 and $j' > k$ held by agent 2. Then $\frac{\Delta_j}{\Delta_{j'}}$ is proportional to $\frac{\lambda_2 \mu_1(\omega_1)}{\lambda_1 \mu_2(\omega_2)}$, which is equal to $\frac{d_k(\omega_2)}{d_k(\omega_1)}$, which is decreasing in k. It then follows from Lemma 13 that the divertibility discount of the tree held by agent 1 increases relative to the one held by agent 2:

$$\frac{\Delta_j}{\Delta_{j'}}(\varepsilon') > \frac{\Delta_j}{\Delta_{j'}}(\varepsilon)$$

for $\varepsilon' > \varepsilon$. In words, when agent 1 becomes a worse pledger for tree ℓ, the divertibility discount of tree ℓ increases and the divertibility discount of all the other trees j held by agent 1 increase by more than that of trees j' held by agent 2. Thus, co-movement in divertibility discount is stronger among assets held by the same type of agents.

4.3.3 Excess return and wealth distribution

We now study the relationship between the initial distribution of wealth, $(\bar{n}_1, \bar{n}_2)$, and equilibrium excess returns. This relationship has received a lot of attention in the recent literature because it is thought to be informative about the impact of shocks to intermediaries' wealth on risk premia. In our model as in the relevant literature, it is natural to identify intermediaries with risk-tolerant agents.

We consider for simplicity the one-tree economy. In this case, the asset pricing formula writes:

$$\frac{\mathbb{E}\left[R(\omega) - R_f\right]}{R_f} = -\mathrm{cov}\left[M(\omega), R(\omega)\right] + \delta\, \mathbb{E}\left[A_i(\omega) R(\omega)\right].$$

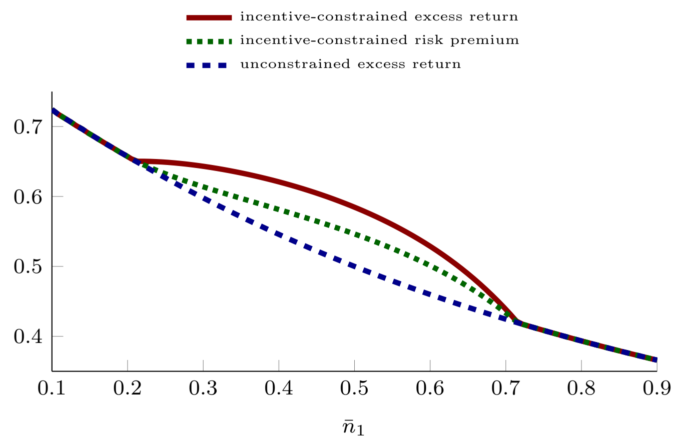

Figure 4: Excess return decomposition. The RRA parameter for agent $i = 1$ is $\gamma_1 = 0.5$, and for agent $i = 2$ it is $\gamma_2 = 1$. There are two equally likely aggregate state, with dividend $d(\omega_1) = 1$ and $d(\omega_2) = 5$.

We divide by R_f so as to normalize the risk-free rate to zero. As we argued earlier, the first term on the right-side is a risk premium, and the second term a divertibility premium. Figure 4 illustrate. The top plain curve is the equilibrium excess return. The bottom dashed curve is the equilibrium excess return in the absence of incentive constraint. The middle dotted curve is the risk premium, as measured by $\mathrm{cov}\left[M(\omega), R(\omega)\right]$. Hence, the distance between the middle dotted curve and the top plain curve is the divertibility premium.

As in the relevant literature there is a monotonically declining relationship between the wealth share of risk-tolerant agents and the excess return on the asset. Differently from the literature, non-linearities arise in an intermediate range of of the distribution of wealth share. This suggests that, if we start from a situation in which risk-tolerant agents are relatively rich, a modest negative shock to the wealth of these agents can lead to sharp rise in excess returns – in the figure, this corresponds to a move from large to intermediate $\bar{n}_1$. In contrast, in the relevant literature, negative shocks to intermediaries wealth have to be large to create non-linearities. As evident from the figure, the rise in excess return is the result of two effects going in the same direction.

First the excess return rises because the pricing kernel $M(\omega)$ becomes more volatile. Namely, in the bad state, the pricing kernel reflects the high marginal utility of the risk-averse agent, who consumes less than in the unconstrained economy because incentive constraints limits the size of insurance payments. Vice versa, in the good state, the pricing kernel reflect the low marginal utility of the risk-averse agent, who consumes more than in the unconstrained economy.

Second, the excess return rises because the divertibility premium increases. But since risk-averse agents consume more in the bad state, risk-tolerant agents must issue larger liabilities and so start facing incentive problems. This increases the shadow incentive cost, reduces the asset price, and correspondingly increases the excess return.

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A Appendix: Proofs

In this appendix we prove all of our results for the generalized model in which δ depends on the agent and (continuously) on the tree type. That is, for each, $i \in I$, the function $j \mapsto \delta_{ij}$ is continuous.

A.1 Proof of Proposition 1

1) First we prove that an equilibrium is incentive constrained Pareto optimal:

Let (c, N) denote an equilibrium allocation with associated price system (q, p). Suppose it is Pareto dominated by some other incentive-feasible allocation $(\hat{c}, \hat{N})$. Then, because utility is strictly increasing, $\hat{c}_i$ must lie strictly outside the budget set of all agents for which $U_i(\hat{c}_i) > U_i(c)i)$. Otherwise, these agents would have a strict incentive to switch to $\hat{c}_i$. Likewise, $\hat{c}_i$ must lie weakly outside the budget set set of all agents for which $U_i(\hat{c}_i) = U_i(c)i)$. Otherwise, these agents would have strict incentive to increase their consumption in some state, which would respect incentive compatibility. Taken together, we obtain:

$$\sum_{\omega \in \Omega} q(\omega)\hat{c}_i(\omega) + \int p_j \, d\hat{N}_{ij} \geq \bar{n}_i \int p_j \, d\bar{N}_j + \int \sum_{\omega \in \Omega} q(\omega) d_j(\omega) \, d\hat{N}_{ij},$$

with one strict inequality for all $i \in I$ such that $U_i(\hat{c}_i) > U_i(c_i)$. Adding up across all agents we obtain that:

$$\sum_{\omega \in \Omega} q(\omega) \left\{ \sum_{i \in I} \hat{c}_i(\omega) - \sum_{i \in I} \int d_j(\omega) \, d\hat{N}_{ij} \right\} + \int p_j \left\{ \sum_{i \in I} d\hat{N}_{ij} - d\bar{N}_j \right\} > 0,$$

which contradicts the feasibility of $(\hat{c}, \hat{N})$.

QED

A.2 Proof of Proposition 2

Our proof of existence proceeds as follows. In Section A.2.1 we define the Planner's Problem, we study some of its elementary properties, and we derive necessary and sufficient optimality conditions for a solution. In Section A.2.2, we turn to the equilibrium and derive first-order necessary and sufficient conditions for a solution to the agent's problem. Comparing the first-order conditions for the Planner and for the agent, in Section A.2.3 we show an equivalence between the set of equilibrium allocations, and the set of solutions to the Planner's problem with zero wealth transfers. We then establish the existence of a solution to the Planner's problem with zero wealth transfer. Omitted proofs are in Supplementary Appendix B.

In what follows we identify any measure with its cumulative distribution function. That is, we identify $\mathcal{M}_+$ with the set of increasing and right-continuous functions over $[0, 1$. We denote by $\mathcal{M}$ the vector space of functions which can be written as $F = F_1 - F_2$, where both F_1 and F_2 belong to $\mathcal{M}_+$. We endow $\mathcal{M}$ with the total variation norm. Given any sequence $N^k \in \mathcal{M}$, we said that N^k *converges strongly* towards N , and write $N^k \to N$, if $\lim_{k\to\infty} \|N^k - N\| = 0$. We say that N^k *converges weakly* towards N, and write $N^k \Rightarrow N$, if $\int f_j \, dN_j^k \to \int f_j \, dN_j$ for all continuous real-valued functions $j \mapsto f_j$ over $[0,1]$. A set of allocations K is said to be *weakly closed* if for any weakly converging sequence $(c^k, N^k) \in K$, i.e. such that $c^k \to c$ and $N^k \Rightarrow N$, then the limit of the sequence belongs to K, i.e., $(c, N) \in K$. The set K is said to be *weakly compact* if for any sequence $(c^k, N^k) \in K$, there exist some subsequence (c^ℓ, N^ℓ) and some $(c, N) \in K$ such that $c^\ell \to c$ and $N^\ell \Rightarrow N$.

A.2.1 The Planner's Problem

Let $\mathcal{A}$ denote the simplex, i.e., the set of welfare weights $\alpha \equiv (\alpha_1, \alpha_2, \ldots, \alpha_I)$ such that $\alpha_i \geq 0$ and $\sum_{i \in I} \alpha_i = 1$. Given any $\alpha \in \mathcal{A}$, and given any $(c, N) \in X$, social welfare is defined as

$$W(\alpha, c, N) \equiv \sum_{i \in I} \alpha_i \sum_{\omega \in \Omega} \pi(\omega) u_i \left[c_i(\omega)\right].$$

In the above formula, when $u_i(0) = -\infty$, we let $\alpha_i u_i \left[c_i(\omega)\right] = 0$ if $\alpha_i = c_i(\omega) = 0$.

Given weight $\alpha \in \mathcal{A}$, the *Planner's Problem* is:

$$W^\star(\alpha) = \sup W(\alpha, c, N) \tag{23}$$

with respect to incentive feasible allocations, i.e., with respect to $(c, N) \in X$ satisfying (5), (7) and (8). We let $\Gamma^\star(\alpha)$ denote the set of allocations solving (23). To show the existence of a solution, we rely on:

Lemma 14 *The set of incentive feasible allocations is weakly compact.*

The proof relies on Helly's Selection Theorem (Theorem 12.9 in Stokey and Lucas (1989)) which allows to extract weakly convergence subsequences from bounded sequences in $\mathcal{M}_+$. The feasibility and incentive compatibility constraints hold in the limit by definition of weak convergence. We add to the argument in Stokey and Lucas (1989) by showing that the feasibility constraint for asset holdings is also satisfied in the limit. With this result in mind, we show in the supplementary appendix:

Proposition 15 *The planner's value $W^\star(\alpha)$ is a continuous function of $\alpha \in \mathcal{A}$, and the maximum correspondence $\Gamma^\star(\alpha)$ is non-empty, weakly compact, convex, and has a weakly closed graph. Moreover, consider any sequence $\alpha^k \to \bar{\alpha}$ and an associated sequence of optimal allocations $(c^k, N^k) \in \Gamma^\star(\alpha^k)$. Then, if $\bar{\alpha}_i = 0$, $\lim_{k\to\infty} \alpha_i^k u' \left[c_i^k(\omega)\right] c_i^k(\omega) = 0$ for all $\omega \in \Omega$.*

If $u_i(0) = 0$ for all $i \in I$, the result follow from the same argument as in the proof of the Theorem of the Maximum (see, for example, Theorem 3.6 in Stokey and Lucas (1989)). If $u_i(0) = -\infty$ for some i, then we need to adapt the argument because the social welfare function is not continuous at (α, c, N) such that $\alpha_i = c_i(\omega) = 0$. Likewise, the result concerning $\alpha_i^k u' \left[c_i^k(\omega)\right] c_i^k(\omega) = 0$ is obvious if $u_i(0) = 0$, but requires some additional work when $u_i(0) = -\infty$.

To compare equilibria with solution of the Planner's Problem, we rely on first-order conditions. We first derive necessary conditions. To do so, we cannot apply the Lagrange multiplier theorems of Luenberger (1969), because they do not accommodate equality constraints. Even if we consider a "relaxed problem" where equality constraints are replaced by inequality constraints, the theorems do not apply because the relevant positive cone has an empty interior. We therefore exploit the structure of the problem to derive first-order conditions by hand. To do so we consider, for any N, the maximized objective with respect to c. We then use an Envelope Theorem of Milgrom and Segal (2002) to explicitly calculate the directional derivative of this maximized objective with respect to N. We obtain:

Proposition 16 *Suppose $(c, N) \in X$ solves the Planner's problem given $\alpha \in \mathcal{A}$. Then there exists multipliers $\hat{q} \in \mathbb{R}_+^{|\Omega|}$ and $\hat{\mu} \in \mathbb{R}_+^{|\Omega| \times |I|}$ such that (c, N) satisfies two sets of conditions.*

- *First-order conditions:*

$$\begin{aligned} \alpha_i \pi(\omega) u_i' \left[c_i(\omega)\right] + \hat{\mu}_i(\omega) &= \hat{q}(\omega), \quad \forall (i, \omega) \in I \times \Omega \\ \int \left[\hat{p}_j - \hat{v}_{ij}\right] dN_{ij} &= 0, \end{aligned}$$

  where $\hat{v}_{ij} \equiv \sum_{\omega \in \Omega} \hat{q}(\omega) d_j(\omega) - \sum_{\omega \in \Omega} \hat{\mu}_i(\omega) \delta_{ij} d_j(\omega)$, and $\hat{p}_j \equiv \max_{i \in I} \hat{v}_{ij}$.

- *Complementary slackness conditions:*

$$\begin{aligned} \hat{q}(\omega) \left[\sum_{i \in I} \int d_j(\omega)\, dN_{ij} - \sum_{i \in I} c_i(\omega)\right] &= 0 \quad \forall \omega \in \Omega \\ \hat{\mu}_i(\omega) \left[c_i(\omega) - \int \delta_{ij} d_j(\omega)\, dN_{ij}\right] &= 0 \quad \forall (i, \omega) \in I \times \Omega. \end{aligned}$$

Although the above conditions are also sufficient, it is convenient to state more general sufficient conditions, where $\hat{p}$ is taken to be some abstract continuous linear functional. This allows to show that any equilibrium is a solution to the Planner's Problem, even if the pricing functional cannot be represented by a continuous function. Then, using the necessary conditions derived in Proposition 16, one can show that the same equilibrium allocation can be supported by a pricing functional represented by a continuous function, establishing the claim in footnote 3.

Proposition 17 *An incentive-feasible allocation $(c, N) \in X$ solves the Planner's problem if there exist multipliers $\hat{q} \in \mathbb{R}_+^{|\Omega|}$, $\hat{\mu} \in \mathbb{R}_+^{|\Omega| \times |I|}$, and a continuous linear functional $\hat{p}$ satisfying the following two sets of conditions.*

- *First-order conditions:*

$$\begin{aligned}
\alpha_i \pi(\omega) u_i' \left[c_i(\omega)\right] + \hat{\mu}_i(\omega) &= \hat{q}(\omega), \quad \forall (i,\omega) \in I \times \Omega \\
\hat{p} \cdot M - \int \hat{v}_{ij} \, dM_{ij} &\geq 0 \quad \forall M_i \in \mathcal{M}_+ \text{ and } i \in I, \text{ with } \text{“}=\text{”} \text{ if } M = N_i,
\end{aligned}$$

where $\hat{v}_{ij} \equiv \sum_{\omega\in\Omega} q(\omega) d_j(\omega) - \sum_{\omega\in\Omega} \mu_i(\omega)\delta_{ij} d_j(\omega)$.

- *Complementary slackness conditions:*

$$\begin{aligned}
\hat{q}(\omega) \left[\sum_{i\in I} \int d_j(\omega)\, dN_{ij} - \sum_{i\in I} c_i(\omega)\right] &= 0 \quad \forall \omega \in \Omega \\
\hat{\mu}_i(\omega) \left[c_i(\omega) - \int \delta_{ij} d_j(\omega)\, dN_{ij}\right] &= 0 \quad \forall (i,\omega) \in I \times \Omega.
\end{aligned}$$

A.2.2 Optimality conditions for the Agent's Problem

Notice that the range of the constraint set in the agent's problem is finitely dimensional. In this case, the "interior point condition" for the positive cone associated with the constraint set is immediately satisfied and so one can apply the general Lagrange multiplier theorems shown in Section 8.3 and 8.3 of Luenberger (1969).

Proposition 18 *A $(c_i, N_i) \in X_i$ solve the agent's problem if and only if it satisfies the intertemporal budget constraint, (6), the incentive compatibility constraint (5), and there exists multipliers $\lambda_i \in \mathbb{R}_+$, $\mu_i \in \mathbb{R}_+^{|\Omega|}$ satisfying the following two sets of conditions:*

- *First-order conditions:*

$$\begin{aligned}
\pi(\omega) u_i' \left[c_i(\omega)\right] + \mu_i(\omega) &= \lambda_i q(\omega) \\
\int (p_j - v_{ij})\, dM_{ij} &\geq 0 \quad \forall M_i \in \mathcal{M}_+, \text{ with } \text{“}=\text{”} \text{ if } M_i = N_i,
\end{aligned}$$

where $v_{ij} \equiv \sum_{\omega\in\Omega} q(\omega) d_j(\omega) - \sum_{\omega\in\Omega} \frac{\mu_i(\omega)}{\lambda_i} \delta_{ij} d_j(\omega)$.

- *Complementary slackness conditions:*

$$\lambda_i \left[\bar{n}_i \int p_j \, d\bar{N}_j + \sum_{\omega\in\Omega} q(\omega) \int d_j(\omega)\, dN_{ij} - \int p_j \, dN_{ij} - \sum_{\omega\in\Omega} q(\omega) c_i(\omega)\right] = 0$$

$$\mu_i(\omega) \left[c_i(\omega) - \int \delta_{ij} d_j(\omega)\, dN_{ij}\right] = 0 \quad \forall \omega \in \Omega.$$

There is one difference between this Proposition and the Theorems shown in Section 8.3 and 8.4 of Luenberger (1969): we are asserting that there exists multipliers such that the first-order condition with respect to $c_i(\omega)$ holds with equality. This follows from the following observation: if $c_i(\omega) = 0$, then the incentive compatibility constraint is binding, in particular $\int \delta_{ij} d_j(\omega)\, dN_{ij} = 0$. Therefore, if we raise $\mu_i(\omega)$ so that the first-order condition holds with equality, we leave the product $\mu_i(\omega) \int \delta_{ij} d_j(\omega)\, dN_{ij} = 0$ unchanged, which implies that $p \cdot N_i - \int v_{ij}\, dN_{ij} = 0$ continues to hold. Finally, since raising $\mu_i(\omega)$ decreases v_{ij}, $p \cdot M_i - \int v_{ij}\, dM_{ij}$ remains positive. Taken together, this means that we can always pick multipliers so that the first-order condition with respect to $c_i(\omega)$ holds with equality.

Finally, the following result provide a simple relationship between the value of the agent's endowment, and the marginal value of his consumption plan. This formula will be useful shortly to formulate the equilibrium fixed-point equation.

Lemma 19 *If $(c_i, N_i) \in X_i$ solves the agent's problem, then*

$$\sum_{\omega \in \Omega} \pi(\omega) u' \left[c_i(\omega)\right] c_i(\omega) = \lambda_i \bar{n}_i \int p_j \, d\bar{N}_j.$$

A.2.3 Existence of a Planner's Solution with Zero Wealth Transfer

By comparing the first-order conditions of the Planner and of the agent, we obtain:

Proposition 20 *An allocation $(c, N) \in X$ is an equilibrium allocation if and only if there exists $\alpha \in \mathcal{A}$ such that:*

- *(c, N) solves the Planner's problem given α;*
- *For all $i \in I$, $\alpha_i \sum_{\omega \in \Omega} \pi(\omega) u_i' \left[c_i(\omega)\right] c_i(\omega) = \bar{n}_i \sum_{k \in I} \sum_{\omega \in \Omega} \pi(\omega) u_k' \left[c_k(\omega)\right] c_k(\omega)$.*

In particular, given a solution of the Planner's problem satisfying the above two conditions, an equilibrium price system is given by the multipliers $(\hat{q}, \hat{p})$ of Proposition 16.

Intuitively, comparing the first-order conditions of the Planner and of the agent reveals that the weight α_i must be proportional to $1/\lambda_i$, the inverse of the Lagrange multiplier on the agent's budget constraint. It then follows from Lemma 19 that, for all agents $i \in I$:

$$\alpha_i \sum_{\omega \in \Omega} \pi(\omega) u_i' \left[c_i(\omega)\right] = \bar{n}_i \times \left[\sum_{k \in I} \frac{1}{\lambda_k}\right]^{-1} \times \int p_j \, d\bar{N}_j.$$

The second condition then follows because $\sum_{i \in I} \bar{n}_i = 1$. The final result about the price system follows from direct comparison of the first-order conditions of the agent and the planner.

We are now ready to establish the existence of an equilibrium. Let $\Delta^\star(\alpha)$ denote the set of transfers:

$$\Delta^\star(\alpha) \equiv \alpha_i \sum_{\omega \in \Omega} \pi(\omega) u_i' \left[c_i(\omega)\right] c_i(\omega) - \bar{n}_i \sum_{k \in I} \alpha_k \sum_{\omega \in \Omega} \pi(\omega) u_k' \left[c_k(\omega)\right] c_k(\omega), \tag{24}$$

generated by all $(c, N) \in \Gamma^\star(\alpha)$, with the convention that $\alpha_i u_i'(c) c = 0$ if $\alpha_i = c = 0$. Using the Kakutani's fixed-point Theorem, as in Negishi (1960) and Magill (1981), we can show:

Proposition 21 *There exists some $\alpha \in \mathcal{A}$, such that $0 \in \Delta^\star(\alpha)$.*

Based on some $\alpha \in \mathcal{A}$, using Proposition 20, we can construct an equilibrium allocation and price system.

A.3 Proof of Proposition 3

Step 1: The equation $0 \in \Delta^\star(\alpha)$ has a unique solution. Since the utility function of agent $i = 2$ is strictly concave, its allocation is uniquely determined in the Planner's problem. But since $c_1(\omega) + c_2(\omega) = \int d_j(\omega)\, d\bar{N}_j$, the consumption allocation of agent 1 is also uniquely determined. Hence $\Delta^\star(\alpha)$, defined in equation (24), is a function and not a correspondence. Moreover since $\Delta_1^\star(\alpha) + \Delta_2^\star(\alpha) = 0$ by construction and $\alpha_1 + \alpha_2 = 1$ by assumption, it is enough to look for a solution of $\Delta_1^\star(\alpha_1, 1 - \alpha_1) = 0$. That is, solving for equilibrium boils down to a one-equation in one-unknown problem. To formulate this problem in simple terms, let

$$\mathrm{MU}_i(c_i) \equiv \sum_\omega \pi(\omega) u_i' \left[c_i(\omega)\right] c_i(\omega).$$

Notice, that with CRRA utility, $\mathrm{MU}_i(c_i) = (1 - \gamma_i) U_i(c_i)$ for $\gamma_i \neq 1$, and $\mathrm{MU}_i(c_i) = 1$ for $\gamma_i = 1$. With this notation, the one-equation-in-one-unknown problem for equilibrium is:

$$\bar{n}_2 \alpha_1 \mathrm{MU}_1(c_1) - \bar{n}_1 \alpha_2 \mathrm{MU}_2(c_2) = 0, \tag{25}$$

where (c_1, c_2) is the consumption allocation chosen by the planner given weight $\alpha \in \mathcal{A}$. We already know from Proposition 21 that this equation has a solution. Our proof of uniqueness is based on the following observation.

Lemma 22 *For any α' and α such that $\alpha_1' > \alpha_1$,*

$$U_1(c_1') \geq U_1(c_1) \text{ and } U_2(c_2') \leq U_2(c_2)$$

$$MU_1(c_1') \geq MU_1(c_1) \text{ and } MU_2(c_2') \leq MU_2(c_2)$$

for all $c \in \Gamma^\star(\alpha)$ and $c' \in \Gamma^\star(\alpha')$.

The proof can be found in the Supplementary Appendix. The inequalities on the first line are intuitive: when the weight on agent 1 increases, then his or her utility increases and that of agent 2 decreases. The inequalities on the second line follows directly because of CRRA utility with coefficient $\gamma_i \in [0, 1]$, which imply that $\mu_i(c) = (1 - \gamma_i) U_i(c)$. With this in mind we go back to the equilibrium equation (25). Let α denote some solution, and consider any $\alpha' \neq \alpha$, for example such that $\alpha_1' > \alpha_1$. Let c and c' denote the consumption allocations associated with α and α'. Then,

$$\begin{aligned}
& \bar{n}_2 \alpha_1' \mathrm{MU}_1(c_1') - \bar{n}_1 \alpha_2' \mathrm{MU}_2(c_2') \\
= \; & \bar{n}_2 \alpha_1' \mathrm{MU}_1(c_1') - \bar{n}_1 \alpha_2' \mathrm{MU}_2(c_2') - \bar{n}_2 \alpha_1 \mathrm{MU}_1(c_1) + \bar{n}_1 \alpha_2 \mathrm{MU}_2(c_2) \\
= \; & \bar{n}_2 \alpha_1' \left[\mathrm{MU}_1(c_1') - \mathrm{MU}_1(c_1)\right] - \bar{n}_1 \alpha_2' \left[\mathrm{MU}_2(c_2') - \mathrm{MU}_2(c_2)\right] + \left(\alpha_1' - \alpha_1\right) \left[\bar{n}_2 \mathrm{MU}_1(c_1) + \bar{n}_1 \mathrm{MU}_2(c_2)\right] > 0.
\end{aligned}$$

In the above, the second line follows from subtracting $\bar{n}_2\alpha_1\mathrm{MU}_1(c_1) - \bar{n}_1\alpha_2\mathrm{MU}_2(c_2) = 0$ since α was assumed to solve (25). The third line follows from re-arranging terms and keeping in mind that $\alpha_1' - \alpha_1 = \alpha_2 - \alpha_2'$. The inequality follows from Lemma 22, and from the fact that marginal utilities are strictly positive. Vice versa, if we consider some $\alpha' \neq \alpha$ such that $\alpha_1' < \alpha_1$, we obtain that the equilibrium equation (25) is strictly negative. Therefore, equation weight, α, has a unique solution.

Step 2: the various uniqueness claims. Consider any equilibrium allocation, (c, N), and price system, (p, q). From Proposition 20, we know that (c, N) solves the Planner's given the unique set of weights such that $\Delta^\star(\alpha) = 0$. But, as argued above, the consumption allocation is uniquely determined in the Planner's problem. Hence, it follows that the equilibrium allocation is uniquely determined in an equilibrium as well. Next, by direct comparison of first-order conditions, one sees that (c, N) solve the first-order conditions of the Planner's problem with weights $\alpha_i = \beta/\lambda_i$, multipliers $\hat{q}(\omega) = \beta q(\omega)$, $\hat{\mu}_i(\omega) = \alpha_i\mu_i(\omega)$, $\hat{v}_{ij} = \beta v_{ij}$ and $\hat{p}_j = \beta p_j$, where λ_i is the Lagrange multiplier on agent's i budget constraint, and $\beta \equiv \left[\sum_{k\in I} 1/\lambda_k\right]^{-1}$. But from the first-order conditions of the Planner's problem, and given that c is uniquely determined, it follows that $\hat{q}(\omega)$, $\hat{\mu}(\omega)$ and $\hat{v}(\omega)$ are uniquely determined as well. Clearly, this implies that the price of Arrow securities, q, and the private asset valuations, v, are uniquely determined up to the multiplicative constant $1/\beta$. Now turning to the price of assets, we note that the first-order condition of the agent's problem imply that $p_j = v_{ij}$ for almost all assets held by i. Since the private valuations are uniquely determined up to the multiplicative constant $1/\beta$, the same property must hold for the the price assets $\bar{N}$-almost everywhere.

QED

A.4 Proof of Lemma 4

The first bullet point follows because of non-satiation: if an asset price were equal to zero, its demand would be infinite for all agents, and the market would not clear.

For the second bullet point, suppose, towards a contradiction, that there is a Borel set $J \in [0,1]$, $\bar{N}(J) > 0$, such that $p_j > \sum_{\omega\in\Omega} q(\omega)d_j(\omega)$ for all $j \in J$. Since $\bar{N}(J) = \sum_{i\in I} N_i(J)$, there exists some i such that $N_i(J) > 0$. Then this agent could increase its utility strictly as follows. He would could scale down his or her holdings of asset $j \in J$ by $1-\varepsilon$, i.e. choose:

$$\hat{N}_{ij} = \int_0^j \left(1 - \mathbb{I}_{\{k\in J\}}\right) dN_{ik},$$

and replace these by an equal amount of financial asset of size ε with identical cash flow, namely $d_j(\omega)$ for each $\omega \in \Omega$. This would create strictly positive profit and so would allow to i increase consumption in all states. This is clearly budget feasible. This also respects the divertibility constraint, since consumption increases and asset holdings decrease. The utility of the agent increases strictly, which contradicts optimality.

QED

A.5 Proof of Lemma 5

The conditions are clearly necessary. Indeed, (9) follows from feasibility, while (10) follows from incentive compatibility. To see that these conditions are also sufficient, we show that the allocation made up of c^0 and N^δ solving (9)-(10), is a $\delta = 0$-equilibrium together with price system (p^0, q^0). Since the feasibility conditions (9) and (10) are verified by construction, we only need to verify optimality. We recall first that, in a $\delta = 0$-equilibrium, no-arbitrage implies that:

$$p_j^0 = \sum_{\omega \in \Omega} q^0(\omega) d_j(\omega).$$

It follows from this no-arbitrage condition that the holding of trees cancel out from both sides of agent i's inter-temporal budget constraint, (6). Hence, for each $i \in I$, (c_i^0, N_i^δ) jointly satisfy the budget constraint (6) given price (p^0, q^0). They also satisfy the incentive compatibility constraint (5) by (10). Since this consumption and tree holdings are optimal for the agent in the absence of the divertibility constraint, they must be optimal with it.

QED

A.6 Proof of Proposition 6

Consider the first bullet point. It follows directly from Lemma 5. Indeed, the second condition (10) of Lemma 5 holds by construction since $\delta_{ij} \in [0, 1)$.

Next, consider the second bullet point. It is well known that, in this case, in a $\delta = 0$-equilibrium, agents have constant consumption share. That is, there exists some $\{\alpha_i\}_{i \in I}$ such that $\sum_{i \in I} \alpha_i = 1$ and $c_i(\omega) = \alpha_i \sum_{j \in J} d_j(\omega)$ for all $i \in I$. One then immediately sees that $n_{ij}^\delta = \alpha_i$ satisfies the two conditions of Lemma 5 for any $\delta > 0$.

QED

A.7 Proof of Proposition 7

As before we state proofs for our results when δ_{ij} is assumed to depend both on the type of agent holding the asset and on the type of the asset. In this case, the Proposition holds under the additional restriction that:

$$\frac{\delta_{1j} d_j(\omega_1)}{\delta_{2j} d_j(\omega_2)}, \tag{26}$$

is strictly increasing. Notice that this restriction is automatically satisfied whenever $\delta_{1j} = \delta_{2j}$ for all j. The generalization of (18)-(19) is

$$c_1(\omega_1) \geq \int_{j\in[0,k)} \delta_{1j} d_j(\omega_1) d\bar{N}_j + \delta_{1k} d_k(\omega_1)\Delta N_1 \tag{27}$$

$$c_2(\omega_2) \geq \int_{j\in(k,1]} \delta_{2j} d_j(\omega_2) d\bar{N}_j + \delta_{2k} d_k(\omega_2)\Delta N_2 \tag{28}$$

The "if" part of the Proposition. Pick the smallest possible k and the largest possible ΔN_2 such that the inequalities (27)-(28). Consider the corresponding asset allocation $N_1 = \Delta N_1 \mathbb{I}_4\{j = k\} + \bar{N}\mathbb{I}_{\{j\in[0,k)\}}$ and $N_2 = \Delta N_2 \mathbb{I}_{\{j=k\}} + \bar{N}\mathbb{I}_{\{j\in(k,1]\}}$. By construction, the incentive constraint of agent $i = 1$ holds in state ω_1, and the incentive constraint of agent $i = 2$ holds in state ω_2. If N allocates all assets to agent $i = 2$, that is if $k = 0$ and $\Delta N_2 = \bar{N}_0$, the incentive constraint of agent $i = 1$ obviously hold in state ω_2. Otherwise, if some assets are allocated to agent $i = 1$, then the incentive constraint of agent $i = 2$ binds in state ω_1. Given $\delta_{ij} < 1$, this implies that the incentive constraint of agent $i = 1$ holds in state ω_2.

The only incentive constraint to check is that of agent $i = 2$ in state ω_1. If it holds, we are done. Otherwise,

$$c_2(\omega_1) < \int_{(k,1]} \delta_{2j} d_j(\omega_1)\, d\bar{N}_j + \delta_{2k} d_{2k} \Delta N_2,$$

and we construct another allocation of tree holdings that is incentive compatible. Indeed, consider the proportional asset allocation that delivers agents $i = 1$ and $i = 2$ their consumption in state ω_2: $\tilde{N}_1 = \frac{c_1(\omega_2)}{y(\omega_2)}\bar{N}$ and $\tilde{N}_2 = \frac{c_2(\omega_2)}{y(\omega_2)}\bar{N}$. By construction, with such proportional allocation, the incentive constraint of both agents hold in state ω_2. Since the consumption share of agent $i = 2$ is strictly larger in state ω_1 than in state ω_2, it it follows that agent $i = 2$ incentive compatibility constraint is slack in state ω_1:

$$c_2(\omega_1) > \frac{y(\omega_1)}{y(\omega_2)} c_2(\omega_2) = \frac{c_2(\omega_2)}{y(\omega_2)} \int d_j(\omega_1) d\bar{N}_j = \int d_j(\omega_1) d\tilde{N}_{2j} > \int \delta_{2j} d_j(\omega_1) d\tilde{N}_{2j},$$

where the first inequality states that the consumption share is larger in state ω_1 than in state ω_2, the first equality follows from rearranging and from the definition of $y(\omega_1)$, the second equality follows from the definition of N_2, and the last inequality follows because $\delta_{2j} < 1$.

Taking stock, for the original allocation N, the incentive compatibility constraints hold in state ω_2 for both $i = 1$ and $i = 2$, and it does not hold for in state ω_1 for agent $i = 2$. For the proportional allocation $\tilde{N}$, the incentive compatibility constraints hold in state ω_2 for both $i = 1$ and $i = 2$, and it is holds with strict inequality in state ω_1 for agent $i = 2$. Therefore, there is a convex combination of N and $\tilde{N}$ such that the incentive compatibility constraint is binding in state ω_1 for agent $i = 2$. This implies that the incentive compatibility constraint holds in state ω_1 for agent $i = 1$. Clearly, the incentive compatibility constraint also hold in state ω_2 for both agents since they hold separately for N and $\tilde{N}$.

The "only if" part of the Proposition. As before, pick the smallest possible k and the largest possible ΔN_2 such that (28) holds. If $k = 0$ and $\Delta N_2 = \bar{N}_0$, then (27) evidently holds. Otherwise, (28) holds with equality and we need to establish that that (27) holds as well. To that end, consider any N such that (c, N) is incentive feasible. Then:

$$\begin{aligned}
& \int_{[0,k)} \delta_{1j} d_j(\omega_1) d\bar{N}_j + \delta_{1k} d_k(\omega_1) \Delta N_1 \\
= \; & \int_{[0,k)} \delta_{1j} d_j(\omega_1) \left(dN_{1j} + dN_{2k}\right) + \delta_{1k} d_k(\omega_1) \Delta N_1 \\
= \; & \int_{[0,1]} \delta_{1j} d_j(\omega_1) dN_{1j} - \int_{[k,1]} \delta_{1j} d_j(\omega_1) dN_{1j} + \delta_{1k} d_k(\omega_1) \Delta N_1 + \int_{[0,k)} \delta_{1j} d_j(\omega_1) dN_{2j} \\
\leq \; & c_1(\omega_1) - \int_{[k,1]} \delta_{2j} d_j(\omega_2) \frac{\delta_{1k} d_k(\omega_1)}{\delta_{2k} d_k(\omega_2)} dN_{1j} + \delta_{1k} d_k(\omega_1) \Delta N_1 + \int_{[0,k)} \delta_{2j} d_j(\omega_2) \frac{\delta_{1k} d_k(\omega_1)}{\delta_{2k} d_k(\omega_2)} dN_{2j} \\
= \; & c_1(\omega_1) + \frac{\delta_{1k} d_k(\omega_1)}{\delta_{2k} d_k(\omega_2)} \left[\int_{[0,1]} \delta_{2j} d_j(\omega_2) dN_{2j} + \delta_{2k} d_k(\omega_2) \Delta N_1 - \delta_{2k} d_k(\omega_2) \left(N_{2k} - N_{2k-}\right) - \int_{(k,1]} \delta_{2j} d_j(\omega_2) d\bar{N}_j \right] \\
= \; & c_1(\omega_1) + \frac{\delta_{1k} d_k(\omega_1)}{\delta_{2k} d_k(\omega_2)} \left[\underbrace{\int_{[0,1]} \delta_{2j} d_j(\omega_2) dN_{2j}}_{\leq c_2(\omega_2)} - \underbrace{\left(\delta_{2k} d_k(\omega_2) \Delta N_2 + \int_{(k,1]} \delta_{2j} d_j(\omega_2) d\bar{N}_j \right)}_{= c_2(\omega_2)} \right] \leq c_1(\omega_1),
\end{aligned}$$

where: the second line follows by feasibility; $\bar{N} = N_1 + N_2$, the third line follows by rearranging and using the assumption that (c, N) is incentive feasible; the fourth line follows by using the condition that (26) is strictly increasing; the fifth line by rearranging and using feasibility again; and the sixth line by our assumption that (c, N) is incentive feasible and by our observation that (19) must hold with equality by our choice of k and ΔN_2.

QED

A.8 Proof of Proposition 8

As for Proposition 7, we offer a proof in the general case when δ_{ij} is assumed to depend both on the identity of the asset holders and on the type of the asset, maintaining the restriction that

$$\frac{\delta_{1j} d_j(\omega_1)}{\delta_{2j} d_j(\omega_2)}, \tag{29}$$

is strictly increasing.

The "if" part follows because, with the proposed asset allocation, the incentive constraint of agent $i = 1$ binds in state ω_1, and that of agent $i = 2$ binds in state ω_2. It then follows that the two other incentive constraints are slack.

For the "only if" part, consider any asset allocation such that (c, N) is incentive feasible. Then the incentive constraint of agent $i = 1$ in state ω_1 writes:

$$c_1(\omega_1) = \int_{[0,k)} \delta_{1j} d_j(\omega_1)\, d\bar{N}_j + \delta_{1k} d_k(\omega_1) \Delta N_1 \geq \int \delta_{1j} d_j(\omega_1) dN_{1j}$$

Using that $d\bar{N}_j = dN_{1j} + dN_{2j}$ we then obtain that:

$$\int_{[0,k)} \delta_{1j} d_j(\omega_1)\, dN_{2j} + \delta_{1k} d_k(\omega_1) \Delta N_1 \geq \int_{(k,1]} \delta_{1j} d_j(\omega_1) dN_{1j} + \delta_{1k} d_k(\omega_1)\left(N_{1k} - N_{1k-}\right) \tag{30}$$

Proceeding analogously with the incentive constraint of agent $i = 2$ in state ω_2, we obtain:

$$\int_{(k,1]} \delta_{2j} d_j(\omega_2)\, dN_{1j} + \delta_{2k} d_k(\omega_2) \Delta N_2 \geq \int_{[0,k)} \delta_{2j} d_j(\omega_2) dN_{2j} + \delta_{2k} d_k(\omega_2)\left(N_{2k} - N_{2k-}\right) \tag{31}$$

Now multiply equation (30) by $\delta_{2k} d_k(\omega_2)$, and equation (31) by $\delta_{1k} d_k(\omega_1)$ and add the two inequalities. The $j = k$ terms drop because, by feasibility, $\Delta N_1 + \Delta N_2 = (N_{1k} - N_{1k-}) + (N_{2k} - N_{2k-})$. We thus obtain:

$$\int_{[0,k)} \delta_{1j} d_j(\omega_1) \delta_{2k} d_j(\omega_2)\, dN_{2j} + \int_{(k,1]} \delta_{2j} d_j(\omega_2) \delta_{1k} d_k(\omega_1)\, dN_{1j} \geq \int_{(k,1]} \delta_{1j} d_j(\omega_1) \delta_{2k} d_k(\omega_2) dN_{1j} + \int_{[0,k)} \delta_{2j} d_j(\omega_2) \delta_{1k} d_k(\omega_1) dN_{2j}.$$

After rearranging:

$$\int_{[0,k)} \left[\delta_{1j} d_j(\omega_1) \delta_{2k} d_j(\omega_2) - \delta_{2j} d_j(\omega_2) \delta_{1k} d_k(\omega_1)\right]\, dN_{2j} \geq \int_{(k,1]} \left[\delta_{1j} d_j(\omega_1) \delta_{2k} d_k(\omega_2) dN_{1j} - \delta_{2j} d_j(\omega_2) \delta_{1k} d_k(\omega_1)\right]\, dN_{1j}$$

But, by (29), the integrand on the left-hand side is strictly negative over $[0, k)$, while the integrand on the right-hand side is strictly positive over $(k, 1]$. Therefore, both integrals are zero, agent $i = 2$ holds no assets in $[0, k)$ and all assets in $(k, 1]$, while agent $i = 1$ holds all assets in $[0, k)$ and no asset in $(k, 1]$. Plugging this back into the incentive compatibility constraint, we can determined the each agent's holdings of asset k. Indeed, we obtain:

$$\delta_{1k} d_k(\omega_1) \Delta N_1 \geq \delta_{1k} d_k(\omega_1)\left(N_{1k} - N_{1k-}\right) \text{ and } \delta_{2k} d_k(\omega_2) \Delta N_2 \geq \delta_{2k} d_k(\omega_1)\left(N_{2k} - N_{2k-}\right).$$

Since $\Delta N_1 + \Delta N_2 = (N_{1k} - N_{1k-}) + (N_{2k} - N_{2k-}) = \bar{N}_k - \bar{N}_{k-}$, it follows that $\Delta N_1 = N_{1k} - N_{1k-}$ and $\Delta N_2 = N_{2k} - N_{2k-}$.

QED

A.9 Proof of Lemma 9

Consider first the first-best allocation, $c^\star$. The first-order condition of the Planner's problem implies

$$\alpha_1 \left[c_1^\star(\omega)\right]^{-\gamma_1} - \alpha_2 \left[y(\omega) - c_1^\star(\omega)\right]^{-\gamma_2} = 0,$$

for all $\omega \in \Omega$. In terms of consumption share, $c(\omega)/y(\omega)$, this equation becomes:

$$\alpha_1 \left[\frac{c_1^\star(\omega)}{y(\omega)}\right]^{-\gamma_1} y(\omega)^{\gamma_2 - \gamma_1} - \alpha_2 \left[1 - \frac{c_1^\star(\omega)}{y(\omega)}\right]^{-\gamma_2} = 0. \tag{32}$$

Since $\gamma_2 > \gamma_1$, this equation is strictly decreasing in the consumption share and strictly increasing in $y(\omega)$. Hence it follows that the consumption share is strictly increasing in $y(\omega)$, i.e., $c_1^\star(\omega_1)/y(\omega_1) < c_1^\star(\omega_2)/y(\omega_2)$. The inequality for $i = 2$ follows directly because consumption shares add up to one.

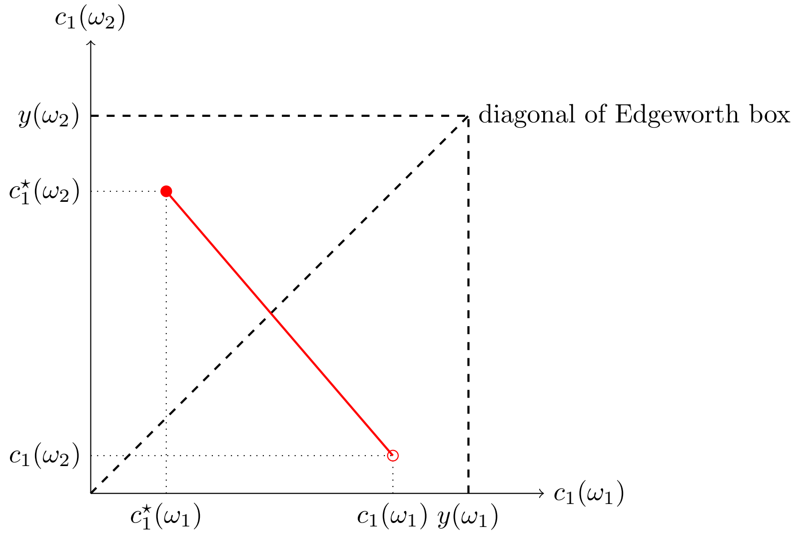

Figure 5: The Edgeworth box for the consumption of agent 1 in state ω_1 (x-axis) and in state ω_2 (y-axis).

Now consider the equilibrium allocation, c. Assume, toward a contradiction, that $c_1(\omega_1)/y(\omega_1) \geq c_1(\omega_2)/y(\omega_2)$, i.e., the consumption shares of agent $i = 1$ lie below the diagonal of the Edgeworth box, as shown in Figure 5. Notice that since the first-best allocation, $c^\star$, satisfies the reverse inequality, it must lie strictly above the diagonal. This implies that $c^\star \neq c$. By strict concavity, social welfare evaluated at c is strictly smaller than social welfare evaluated at $c^\star$, and strictly smaller than social welfare at any point on the segment $(c, c^\star]$ linking c to $c^\star$, shown in red on the figure. Clearly, the segment $[c, c^\star)$ crosses the diagonal at some point $c^\dagger$, which may be c. Since $c^\dagger$ keeps the agent's consumption share constant across states, it can be made incentive feasible by giving agents the corresponding "proportional" asset allocation, i.e., a share in the market portfolio equal to their respective consumption share, $N_i^\dagger = c_i^\dagger(\omega_i)/y(\omega_i)\,\bar{N}$. But since $\delta < 1$, it follows that all incentive constraints are slack for $(c^\dagger, N^\dagger)$. Therefore, points on the segment $(c, c^\star]$ near $c^\dagger$ are incentive feasible as well. But they improve social welfare strictly relative to c, which is a contradiction.

QED

A.10 Proof of Corollary 11

With two agents, the zero-transfer equation (24) writes:

$$\bar{n}_2 \alpha_1 \mathbb{E}\left\{u_1'\left[c_1(\omega)\right] c_1(\omega)\right\} = \bar{n}_1 \alpha_2 \mathbb{E}\left\{u_2'\left[c_2(\omega)\right] c_2(\omega)\right\}$$

With CRRA utility, this can be simplified further:

$$\bar{n}_2 \alpha_1 \mathbb{E}\left[c_1(\omega)^{1-\gamma_1}\right] = \bar{n}_1 \alpha_2 \mathbb{E}\left[c_2(\omega)^{1-\gamma_2}\right],$$

so that:

$$\frac{\bar{n}_1}{\bar{n}_2} = \frac{\alpha_1 \mathbb{E}\left[c_1(\omega)^{1-\gamma_1}\right]}{\alpha_2 \mathbb{E}\left[c_2(\omega)^{1-\gamma_2}\right]}.$$

Now notice that, as α_1/α_2 increases, the solution of the Planner's problem moves to the northeast of the incentive-constrained Pareto set (see Lemma 22 in the Proof of Proposition 3). Clearly, this implies a strictly increasing relationship between $\bar{n}_1/\bar{n}_2$ and α_1/α_2.

QED

A.11 Proof of Lemma 13

Notice that, since the function ϕ_ℓ is the same for both agents, we have that $\delta_{1j} d_j(\omega_1)/\delta_{2j} d_j(\omega_2) = d_j(\omega_1)/d_j(\omega_2)$ is strictly increasing, so all our results apply.

The equilibrium is uniquely pinned down by a two-equation-in-two-unknown problem, for the ratio of the two budget constraints multipliers, $r = \frac{\lambda_1}{\lambda_2}$ and the threshold k determining asset ownership. To obtain the first equation, first note that the continuity of $j \mapsto (\delta_{1j} d_j(\omega_1))/(\delta_{2j} d_j(\omega_2))$ implies that for the threshold asset, the first-order condition with respect to asset holdings holds with an equality for both agents. Thus:

$$F(r,k) \equiv \mu_1(\omega_1)\delta_{1k} d_k(\omega_1) - r\mu_2(\omega_2)\delta_{2k} d_k(\omega_2) = 0. \tag{33}$$

where, from the first-order conditions we have that

$$\begin{aligned}
\mu_1(\omega_1) &= r\pi(\omega_1) u_2' \left[\int_0^1 \left(1 - \delta_{1j}\mathbb{I}_{\{j<k\}}\right) d_j(\omega_1)\, d\bar{N}_j\right] - \pi(\omega_1) u_1' \left[\int_0^1 \delta_{1j}\mathbb{I}_{\{j<k\}} d_j(\omega_1)\, d\bar{N}_j\right] \\
\mu_2(\omega_2) &= \frac{1}{r}\pi(\omega_2) u_1' \left[\int_0^1 \left(1 - \delta_{2j}\mathbb{I}_{\{j\geq k\}}\right) d_j(\omega_2)\, d\bar{N}_j\right] - \pi(\omega_2) u_2' \left[\int_0^1 \delta_{2j}\mathbb{I}_{\{j\geq k\}} d_j(\omega_2)\, d\bar{N}_j\right].
\end{aligned}$$

Notice that the continuity of the distribution of asset supplies mean that the allocation of the supply of threshold assets between agents is irrelevant. The second equilibrium equation is (24) which here takes the form:

$$G(r,k) \equiv \mathbb{E}[u_1'(c_1(\omega))c_1(\omega)] - r\frac{\bar{n}_1}{\bar{n}_2}\mathbb{E}[u_2'(c_2(\omega))c_2(\omega)] = 0, \tag{34}$$

where $c_1(\omega_1) = \int_0^k \delta_{1j} d_j(\omega_1)\, d\bar{N}_j$, $c_2(\omega_1) = \int_0^1 d_j(\omega_1)\, d\bar{N}_j - c_1(\omega_1)$, $c_2(\omega_2) = \int_k^1 \delta_{2j} d_j(\omega_2)\, d\bar{N}_j$, and $c_1(\omega_2) = \int_0^1 d_j(\omega_2)\, d\bar{N}_j - c_2(\omega_2)$.

The function $F(r,k)/(\delta_{2k} d_k(\omega_2))$ is strictly increasing and continuous in both r and k. Moreover, one can explicitly solve for r as a function of k, $\rho(k)$. This function is strictly decreasing and, because of the Inada condition $u_i'(0) = +\infty$, goes to infinity as k goes to zero, $\lim_{k\to 0}\rho(k) = \infty$, and goes to zero as k goes to one, $\lim_{k\to 1}\rho(k) = 0$.

Since $\bar{N}_j$ is strictly increasing, it follows that both $c_1(\omega_1)$ and $c_1(\omega_2)$ are strictly increasing in k while both $c_2(\omega_1)$ and $c_2(\omega_2)$ are strictly decreasing in k. Recall that the coefficient of relative risk aversion are both less than one,

$0 \leq \gamma_1 < \gamma_2 \leq 1$. Therefore, the function $G(r,k)$ is strictly decreasing in r and strictly increasing in k. Plugging in the function $\rho(k)$ defined above, we obtain a strictly increasing function $k \mapsto G(\rho(k),k)$. Given our earlier observation that $\lim_{k\to 0}\rho(k) = \infty$ and $\lim_{k\to 1}\rho(k) = 0$, it follows that $k \mapsto G(\rho(k),k)$ is strictly negative when $k \simeq 0$, and strictly positive when $k \simeq 1$. Thus, the equilibrium threshold is the unique solution of $G(\rho(k),k) = 0$. Clearly $c_1(\omega_1)$ increases with ε, while $c_2(\omega_2)$ stays the same. This implies that $\rho(k)$ shifts down, and that $G(\rho(k),k)$ shifts down as well. Hence $k(\varepsilon') < k(\varepsilon)$ if $\varepsilon' > \varepsilon$.

$$\frac{dk}{d\varepsilon} < 0.$$

QED

B Supplementary appendix

B.1 Proof of Lemma 14

For this proof, in order to apply some of the results in Chapter 12 of Stokey and Lucas (1989), we extend measures $M \in \mathcal{M}_+$ to the entire real line, $\mathbb{R}$, by setting $M_j = 0$ for all $j < 0$, and $M_j = M_1$ for all $j \geq 1$. Now consider a sequence (c^k, N^k) of incentive feasible allocation. Given that c^k belongs to a finite dimensional space and is bounded, it has a converging subsequence. Given that $\sum_{i \in I} N_i^k = \bar{N}$, N_{ij} is bounded above by $\bar{N}_j$ for all $(i,j) \in I \times \mathbb{R}$, an an application of Helly's selection Theorem (Theorem 12.9 in Stokey and Lucas (1989) extended to finitE measure instead of distribution) shows that for each $i \in I$, N_i^k has a subsequence such that N_i^ℓ converging weakly in $\mathcal{M}_+$. Taken together, this means that there exists a subsequence (c^ℓ, N^ℓ) of (c^k, N^k) and some $(c, N) \in X$ such that $c^\ell \to c$ and $N_i^\ell \Rightarrow N_i$ for each $i \in I$.

What is left to show is that (c, N) is incentive feasible. Given that $j \mapsto d_j(\omega)$ and $j \mapsto \delta_{ij}$ are continuous, the definition of weak convergence allows us to assert that, since the feasibility constraint for consumption, (7), and in the incentive compatibility constraints, (5), hold for each (c^ℓ, N^ℓ), then it must also hold in the limit for (c, N). The only difficulty is to show that the feasibility constraint for holdings is also satisfied. For this we rely on the characterization of weak convergence provided by Theorem 12.8 in Stokey and Lucas (1989), easily extended to bounded measures. It asserts that N_i^ℓ converges pointwise at each continuity point of their limit, N_i. Therefore, for any $j \in \mathbb{R}$ such that all $(N_i)_{i \in I}$ are continuous, we have:

$$\sum_{i \in I} N_{ij}^\ell \to \sum_{j \in I} N_{ij}.$$

But recall that the feasibility constraint for holdings is satisfied for each j: $\sum_{i \in I} N_{ij}^\ell = \bar{N}_j$. Together with the above, this implies that

$$\sum_{i \in I} N_{ij} = \bar{N}_j,$$

for all $j \in \mathbb{R}$ such as all $(N_i)_{i \in I}$ are continuous. Now recall that N_i are increasing functions, and so have countably many discontinuity points. This implies that for any $j \in \mathbb{R}$, there is a sequence of $j_n \downarrow j$ such that j_n is a continuity point for all $(N_i)_{i \in I}$. Hence, for all j_n, we have

$$\sum_{i \in I} N_{ij_n} = \bar{N}_{j_n}.$$

Since $j \mapsto N_{ij}$ and $\bar{N}_j$ are all right continuous functions we can take the limit and obtain that $\sum_{i \in I} N_{ij} = \bar{N}_j$ for all $j \in \mathbb{R}$, as required.

B.2 Proof of Proposition 15

In all what follow we let:

$$y(\omega) \equiv \sum_{j \in J} d_j(\omega), \ \underline{y} \equiv \min_{\omega \in \Omega} y(\omega), \ \text{ and } \bar{y} \equiv \max_{\omega \in \Omega} y(\omega).$$

Proof that $\Gamma^\star(\alpha)$ is not empty. We first show that the supremum is achieved. The only difficulty with this proof arises when $\alpha_i > 0$ and $u_i(0) = -\infty$ for some $i \in I$, because in this case the objective is not continuous when $\alpha_i = c_i = 0$. However, in the planner's problem, one can restrict attention to $c_i(\omega)$ that are bounded away from zero. To see this, we first note that $c_i(\omega) = y(\omega)/I$ is feasible, implying that:

$$W^\star(\alpha) \geq \sum_{i \in I} \alpha_i \sum_{\omega \in \Omega} \pi(\omega) u_i \left[y(\omega)/I \right] \geq \sum_{i \in I} \min\{ u_i \left[\underline{y}/I \right], 0 \} \equiv \underline{W}.$$

Also, for each i such that $\alpha_i > 0$ and $u_i(0) = -\infty$, we have that

$$W(\alpha, c, n) \leq \alpha_i \pi(\omega) u_i \left[c_i(\omega) \right] + \sum_{k \neq i} \alpha_k \max\{ u_i(\bar{y}/I), 0\}.$$

Now consider the equation

$$\alpha_i \pi(\omega) u_i \left[c_i(\omega) \right] + \sum_{k \neq i} \alpha_k \max\{ u_i(\bar{y}/I), 0\} = \underline{W}.$$

Since $u_i(0) = -\infty$, the left-hand side is smaller than the right-hand side when $c \to 0$. Since $\underline{W} \leq 0$ by construction, the left-hand side is larger than the right-hand side when $c \to \infty$. Given the strict monotonicity of $u_i(c)$, it follows that the equation has a unique solution, which is decreasing and continuous in α_i. Let $\underline{c}_i(\alpha_i)$ be half of the minimum of these solutions across all $\omega \in \Omega$. By construction, for all allocation (c, n) such that $c_i(\omega) < \underline{c}_i$ for some $\omega \in \Omega$, $W(\alpha, c, n) < \underline{W}$. If we let $\underline{c}_i(\alpha_i) = 0$ for other i, that is for $i \in I$ such that $\alpha_i = 0$ or $u_i(0) = 0$, then, in the Planner's problem, one can restrict attention to allocation such that $c_i(\omega) \geq \underline{c}_i(\alpha_i)$, which we write as $c \geq \underline{c}(\alpha)$. Notice that, by construction, the objective of the planner is continuous over $c \geq \underline{c}(\alpha)$.

Now to show that there is a solution consider any sequence (c^k, N^k) of incentive-feasible allocation such that $W(\alpha, c^k, N^k) \to W^\star(\alpha)$. From the above remark we can focus on a sequence such that $c^k \geq c(\alpha)$. Now Lemma 14, there exists some incentive feasible allocation (c, N) and a subsequence (c^ℓ, N^ℓ) such that $c^\ell \to c$ and $N^\ell \to N$. Going to the limit in the Planner's objective, we obtain that $W(\alpha, c, N) = W^\star(\alpha)$.

Proof that $\Gamma^\star(\alpha)$ is weakly compact. The argument is the same as in the last paragraph, except that we now consider a sequence $(c^k, N^k) \in \Gamma^\star(\alpha)$.

Proof that $\Gamma^\star(\alpha)$ convex-valued. This follows because the objective is concave and the constraints linear.

Proof that $W^\star(\alpha)$ is continuous and $\Gamma^\star(\alpha)$ has a weakly closed graph. Consider any $\bar\alpha \geq 0$ such that $\sum_{i\in I}\bar\alpha_i = 1$ and any sequence $\alpha^k \to \bar\alpha$ and an associated sequence $(c^k, N^k) \in \Gamma^\star(\alpha^k)$. Without loss of generality for this proof, assume that $W^\star(\alpha^k)$ converges to some limit,[11] and that (c^k, N^k) converges weakly towards some incentive feasible allocation (c, N).[12] We want to show that $W^\star(\alpha^k) \to W^\star(\alpha)$ and that $(c, N) \in \Gamma^\star(\alpha)$. Let $I_0 = \{i \in I : \alpha_i = 0 \text{ and } u_i(0) = -\infty\}$. We have:

$$W^\star(\alpha^k) = \sum_{i\notin I_0}\alpha_i^k\sum_{\omega\in\Omega}\pi(\omega)u_i\left[c_i^k(\omega)\right] + \sum_{i\in I_0}\alpha_i^k\sum_{\omega\in\Omega}\pi(\omega)u_i\left[c_i^k(\omega)\right]. \tag{35}$$

By our maintained assumptions, both left-hand side and the first term on the right-hand side have a limit as $k \to \infty$. Hence, the second term on the right-hand side has a limit as well. We argue that this limit must be negative. Indeed, for $i \in I_0$, if $\lim c_i^k(\omega) > 0$, then $\lim \alpha_i^k u_i\left[c_i^k(\omega)\right] = 0$. If $\lim c_i^k(\omega) = 0$, then $\alpha_i^k u_i\left[c_i^k(\omega)\right] < 0$ for k large enough. Hence,

$$\lim\sum_{i\in I_0}\alpha_i^k\sum_{\omega\in\Omega}\pi(\omega)u_i\left[c_i^k(\omega)\right] \leq 0.$$

Therefore:

$$\lim W^\star(\alpha^k) \leq \sum_{i\notin I_0}\bar\alpha_i\sum_{\omega\in\Omega}\pi(\omega)u_i\left[\lim c_i^k(\omega)\right] \leq W^\star(\bar\alpha), \tag{36}$$

since $(\lim c^k, \lim N^k)$ is incentive feasible.

To show the reverse inequality, for all $i \in I_0$, choose some $\phi_i > 0$ such that $\phi_i(\gamma_i - 1) < 1$, where $\gamma_i > 1$ is the assumed CRRA bound for $u_i(c)$. Let $\beta(\alpha) \equiv \sum_{i\in I_0}(\alpha)^{\phi_i(\gamma_i-1)}$. Since $\lim\alpha_i^k = 0$ for all $i \in I_0$, we have that $\lim\beta(\alpha^k) = 0$, hence $\beta(\alpha^k) < 1$ for all k large enough. Given some $(\bar c, \bar n) \in \Gamma^\star(\bar\alpha)$, consider the allocation obtained by scaling down the consumption and asset holding of $i \notin I_0$ by $1 - \beta(\alpha^k)$, and by giving to $i \in I_0$ a consumption equal to $y(\omega)\left(\alpha_i^k\right)_i^\phi$ and an asset allocation equal to a fraction $\left(\alpha_i^k\right)^{\phi_i}$ of the market portfolio. One easily sees that this allocation is incentive feasible. Hence, we have that:

$$W^\star(\alpha^k) \geq \sum_{i\notin I_0}\alpha_i\sum_{\omega\in\Omega}\pi(\omega)u_i\left[\bar c_i(\omega)(1-\beta(\alpha))\right] + \sum_{i\in I_0}\alpha_i\sum_{\omega\in\Omega}\pi(\omega)u_i\left[y(\omega)\alpha_i^{\phi_i}\right].$$

The first term converges to $W^\star(\bar\alpha)$. Using the assumed CRRA bound, $0 < |u(c)| < |K|c^{1-\gamma_i}$ for c close to zero, one sees that the second term goes to zero: indeed $\alpha_i^k|u_i\left[y(\omega)\left(\alpha_i^k\right)^{\phi_i}\right]|$ is bounded above by $|K|y(\omega)^{1-\gamma_i}\left(\alpha_i^k\right)^{1+(1-\gamma_i)\phi_i}$, which goes to zero since $\lim\alpha_i^k = 0$ and ϕ_i was chosen so that $1 + \phi_i(1-\gamma_i) > 0$. Hence, we obtain that $\lim W^\star(\alpha^k) \geq W^\star(\bar\alpha)$. Taken together we have that

$$\lim W^\star(\alpha^k) \geq \sum_{i\notin I_0}\bar\alpha_i\sum_{\omega\in\Omega}\pi(\omega)u_i\left[\lim c_i^k(\omega)\right] = W^\star(\bar\alpha). \tag{37}$$

[11] Indeed, since $W^\star(\alpha)$ is bounded below by $\underline{W}$ and is clearly bounded above, to show convergence towards $W^\star(\alpha)$ it is sufficient to show that every convergent subsequence of $W^\star(\alpha^k)$ converges towards $W^\star(\alpha)$.

[12] From Lemma 14, we can always find a convergence subsequence with this property.

Taken together, (36) and (37) imply that

$$\lim W^\star(\alpha^k) = \sum_{i \notin I_0} \alpha_i \sum_{\omega\in\Omega} u_i \left[\lim c_i^k(\omega)\right] = W(\bar{\alpha}) \text{ and } \lim \sum_{i\in I_0} \alpha_i^k \sum_{\omega\in\Omega} \pi(\omega) u_i \left[c_i^k(\omega)\right] = 0.$$

This establishes that $W^\star(\alpha)$ is continuous and that $\Gamma^\star(\alpha)$ has a closed graph.

Proof that $\lim \alpha_i^k u' \left[c_i^k(\omega)\right] c_i^k(\omega) = 0$ **if** $\lim \alpha_i^k = 0$**.** Consider any sequence $\alpha^k \to \bar{\alpha}$ and any associated sequence (not necessarily converging) (c^k, N^k). Since we have shown that $\Gamma^\star(\alpha)$ has a weakly closed graph, it follows that any converging subsequence of (c^k, N^k) has a limit belonging to $\Gamma^\star(\bar{\alpha})$. But this limit is such that $c_i^k(\omega) = 0$ for all i such that $\bar{\alpha}_i = 0$. Hence, for all i such that $\bar{\alpha}_i = 0$, $\lim c_i^k(\omega) = 0$. If $u_i(0) = 0$, then the result that $\lim \alpha_i^k u' \left[c_i^k(\omega)\right] c_i^k(\omega) = 0$ follows from the inequality $0 \leq u_i'(c)c \leq u_i(c)$.

If $u_i(0) = -\infty$, we need a different argument. Write $W^\star(\alpha^k) = W_1^k + W_2^k$, where

$$W_1^k \equiv \sum_{i\notin I_0} \alpha_i \sum_{\omega\in\Omega} \pi(\omega) u_i \left[c_i^k(\omega)\right] \text{ and } W_2^k \equiv \sum_{i\in I_0} \alpha_i \sum_{\omega\in\Omega} \pi(\omega) u_i \left[c_i^k(\omega)\right].$$

By assumption, we have that $\lim \left(W_1^k + W_2^k\right) = W^\star(\bar{\alpha})$. Notice that W_1^k is bounded. Indeed, it is clearly bounded above because the constraint set is bounded. It is bounded below because, for any $i \notin I_0$ such that $u_i(0) = -\infty$, $\bar{\alpha}_i > 0$ and so α_i^k and hence $\bar{c}_i(\alpha_i^k)$ is bounded away fro zero for k large enough. Given boundedness, we can extract some convergent subsequence W_1^ℓ of W_1^k. Since consumption and asset holdings are incentive feasible, it follows from Lemma 14 that there exists a weakly convergent subsequence (c^p, N^p) of (c^ℓ, N^ℓ). Clearly, $\lim W_1^p = \lim W_1^\ell$. But, using the results of the previous paragraph, we have that $\lim W_1^p = W^\star(\bar{\alpha})$. Hence all convergent subsequences of W_1^k have the same limit $W^\star(\bar{\alpha})$, implying that $\lim W_1^k = W^\star(\bar{\alpha})$ and that $\lim W_2^k = 0$. It follows that, for all k large enough, all terms in W_1^k are negative. Hence, for k large enough, we that for all $i \in I_0$, $W_2^k \leq \alpha_i^k \pi(\omega) u_i \left[c_i^k(\omega)\right] \leq 0$. Since $\lim W_2^k = 0$, it follows that $\lim \alpha_i^k \pi(\omega) u_i \left[c_i^k(\omega)\right] = 0$ as well. The result then follows from the CRRA bound $0 \leq u_i'(c)c \leq \gamma_i |u_i(c)|$.

B.3 Proof of Proposition 16

Fix any feasible N and let:

$$W(\alpha \,|\, N) = \max \sum_{i\in I} \alpha_i U_i(c_i)$$

with respect to $c \in X$, and subject to

$$\sum_{i\in I} c_i(\omega) \leq \sum_{i\in I} \int d_j(\omega) dN_{ij} \quad \forall \omega \in \Omega$$

$$c_i(\omega) \geq \int \delta_{ij} d_j(\omega) dN_{ij} \quad \forall (i,\omega) \in I \times \Omega.$$

From Corollary 28.3 in Rockafellar (1970), $c \in X$ is an optimal solution only if there exists multipliers $\hat{q} \in \mathbb{R}_+^{|\Omega|}$ and $\hat{\mu} \in \mathbb{R}_+^{|\Omega| \times |I|}$ such that:

$$\begin{aligned}
\alpha_i \frac{\partial U_i}{\partial c_i(\omega)} + \hat{\mu}_i(\omega) &\leq \hat{q}(\omega) \\
\hat{q}(\omega) \left[\sum_{i \in I} \int d_j(\omega) dN_{ij} - \sum_{i \in I} c_i(\omega) \right] &= 0, \quad \forall \omega \in \Omega \\
\hat{\mu}_i(\omega) \left[c_i(\omega) - \int \delta_{ij} d_j(\omega) dN_{ij} \right] &= 0, \quad \forall (i, \omega) \in I \times \Omega.
\end{aligned}$$

Notice that we can always choose multipliers such that the first-order condition with respect to $c_i(\omega)$ holds with equality. Indeed, if it holds with a strict inequality for some $\hat{\mu}_i(\omega)$ and some (i, ω), then $c_i(\omega) = 0$ and so the incentive constraint holds with equality. So increasing $\hat{\mu}_i(\omega)$ leaves the complementary slackness conditions unchanged.

Now consider any other feasible $\hat{N} \in \mathcal{M}_+$. Clearly, for any $h \in [0, 1]$, $(1-h)N + h\hat{N} = N + h(\hat{N} - N)$ is also feasible. In the optimization problem $W(\alpha \,|\, N + h \left[\hat{N} - N \right])$, the derivative of the Lagrangian with respect to h, evaluated at $h = 0$, is

$$\begin{aligned}
L_h &= \sum_{i \in I} \int \left[\sum_{\omega \in \Omega} \hat{q}(\omega) d_j(\omega) - \sum_{\omega \in \Omega} \hat{\mu}_i(\omega) \delta_{ij} d_j(\omega) \right] \left[d\hat{N}_{ij} - dN_{ij} \right] \\
&= \sum_{i \in I} \int \hat{v}_{ij} \left[d\hat{N}_{ij} - dN_{ij} \right],
\end{aligned}$$

where, for any set of Lagrange multipliers, $v_{ij} \equiv \sum_{\omega \in \Omega} \hat{q}(\omega) d_j(\omega) - \sum_{\omega \in \Omega} \hat{\mu}_i(\omega) \delta_{ij} d_j(\omega)$. Notice that $\hat{q}(\omega)$ is uniquely determined[13] but $\hat{\mu}_i(\omega)$ may not, when $c_i(\omega) = 0$. One easily sees in particular that any

$$0 \leq \hat{\mu}_i(\omega) \leq \hat{q}(\omega) - \alpha_i \frac{\partial U_i}{\partial c_i(\omega)}$$

solves the first-order conditions. Let $\hat{V}_{ij}$ denote the corresponding interval of $\hat{v}_{ij}$. It follows from Corollary 5 in Milgrom and Segal (2002) that the right-derivative of $W\left(\alpha \,|\, N + h \left[\hat{N} - N \right] \right)$ at $h = 0$ is

$$\frac{d}{dh} W\left(\alpha \,|\, N + h \left[\hat{N} - N \right] \right) \bigg|_{h=0^+} = \min_{\hat{v}_{ij} \in \hat{V}_{ij}} \sum_{i \in I} \int \hat{v}_{ij} \left[d\hat{N}_{ij} - dN_{ij} \right].$$

Now notice that $\int \hat{v}_{ij} dN_{ij}$ does not depend on the particular choice of $\hat{v}_{ij}$. Indeed, whenever $\hat{v}_{ij}$ is not uniquely determined, it is because $c_i(\omega) = 0$ for some $\omega \in \Omega$. But from the incentive compatibility constraint, it then follows that $\int \delta_{ij} d_j(\omega)\, dN_{ij} = 0$, and so $\hat{\mu}_i(\omega) \int \delta_{ij} d_j(\omega)\, dN_{ij} = 0$ as well. Since $\hat{N}_{ij}$ is a positive measure, $\int \hat{v}_{ij} d\hat{N}_{ij}$ is minimized

[13] Indeed for any $\omega \in \Omega$, consider any $i \in I$ such that the incentive compatibility constraint does not bind. Then $c_i(\omega) > 0$ and so the first-order condition holds with equality. If $u_i(c)$ is linear, then $\alpha_i \partial U_i / \partial c_i(\omega) = \alpha_i$ is uniquely determined. If $u_i(c)$ is strictly concave, then $c_i(\omega)$ is uniquely determined and so is $\alpha_i \partial U_i / \partial c_i(\omega)$. Using the first-order condition, it then follows that $\hat{q}(\omega)$ is uniquely determined.

when $\hat{v}_{ij}$ is smallest, which occurs when $\hat{\mu}_i(\omega)$ is largest, that is, when it is chosen so that the first-order condition with respect to $c_i(\omega)$ holds with equality.

Taken together, we obtain that a necessary condition for a feasible N to be optimal is that:

$$\sum_{i\in I}\int \hat{v}_{ij}\left[d\hat{N}_{ij}-dN_{ij}\right]\leq 0, \tag{38}$$

for all feasible $\hat{N}$, where $\hat{v}_{ij}=\sum_{\omega\in\Omega}\hat{q}(\omega)d_j(\omega)-\sum_{\omega\in\Omega}\hat{\mu}_i(\omega)\delta_{ij}d_j(\omega)$ and $\hat{\mu}_i(\omega)$ is chosen so that the first-order condition with respect to $c_i(\omega)$ holds with equality. The proof is concluded by the following Lemma:

Lemma 23 *Condition (38) holds if and only if* $\int\left[\max_{k\in I}\hat{v}_{kj}-\hat{v}_{ij}\right]dN_{ij}=0$ *for all* $i\in I$.

For necessity, consider the correspondence $\Gamma(j)\equiv\arg\max_{k\in I}\hat{v}_{kj}$. By the Measurable Selection Theorem (Theorem 7.6 in Stokey and Lucas (1989)), there exists a measurable selection $\gamma(j)$. Consider then the asset allocation:

$$\hat{N}_{ij}=\int_0^j \mathbb{I}_{\{\gamma(k)=i\}}d\bar{N}_k,$$

which gives the supply of asset k to one agent with the highest valuation, $v_{\gamma(k)k}$. Condition (38) implies that:

$$\begin{aligned}0\geq\sum_{i\in I}\int\hat{v}_{ij}\left[d\hat{N}_{ij}-dN_{ij}\right] &= \sum_{i\in I}\int\hat{v}_{ij}\mathbb{I}_{\{\gamma(j)=i\}}\,d\bar{N}_j-\sum_{i\in I}\hat{v}_{ij}\,dN_{ij}\\ &= \int\max_{k\in I}\hat{v}_{kj}d\bar{N}_j-\int\hat{v}_{ij}dN_{ij}\\ &= \int\left(\max_{k\in I}\hat{v}_{kj}-\hat{v}_{ij}\right)dN_{ij},\end{aligned}$$

where the last equality follows because $\bar{N}=\sum_{i\in I}N_i$. But each term in the sum is positive since $\max\hat{v}_{kj}-\hat{v}_{ij}\geq 0$. It thus follows that each term in the sum is zero, and we are done.

For sufficiency, write

$$\begin{aligned}\sum_{i\in I}\hat{v}_{ij}\left[d\hat{N}_{ij}-dN_{ij}\right] &= \sum_{i\in I}\hat{v}_{ij}\,d\hat{N}_{ij}-\sum_{i\in I}\int\max_{k\in I}v_{kj}dN_{ij}\\ &= \sum_{i\in I}\hat{v}_{ij}\,d\hat{N}_{ij}-\int\max_{k\in I}v_{kj}d\bar{N}_j\\ &= \sum_{i\in I}\left[\hat{v}_{ij}-\max_{k\in I}v_{kj}\right]d\hat{N}_{ij}\leq 0.\end{aligned}$$

where the last equality follows because $\hat{N}$ is feasible.

B.4 Proof of Proposition 17

Consider any (c, N) and multipliers $\hat{q}$, $\hat{\mu}$ and $\hat{p}$ satisfying the first-order conditions in the Proposition. Now let $(\hat{c}, \hat{N}$ denote any other feasible allocation. We have:

$$
\begin{aligned}
& \sum_{i \in I} \alpha_i U_i(c_i) - \sum_{i \in I} \alpha_i U_i(\hat{c}_i) \\
\geq \quad & \sum_{i \in I} \sum_{\omega \in \Omega} \frac{\partial U_i}{\partial c_i(\omega)} \left[c_i(\omega) - \hat{c}_i(\omega)\right] = \sum_{i \in I} \sum_{\omega \in \Omega} \left[\hat{q}(\omega) - \hat{\mu}_i(\omega)\right] \left[c_i(\omega) - \hat{c}_i(\omega)\right] \\
= \quad & \sum_{\omega \in \Omega} \hat{q}(\omega) \left[\sum_{i \in I} c_i(\omega) - \sum_{i \in I} \int d_j(\omega)\, dN_{ij}\right] - \sum_{\omega \in \Omega} \hat{q}(\omega) \left[\sum_{i \in I} \hat{c}_i(\omega) - \sum_{i \in I} \int d_j(\omega)\, d\hat{N}_{ij}\right] \\
& - \sum_{i \in I} \sum_{\omega \in \Omega} \hat{\mu}_i(\omega) \left[c_i(\omega) - \int \delta_{ij} d_j(\omega) dN_{ij}\right] + \sum_{i \in I} \sum_{\omega \in \Omega} \hat{\mu}_i(\omega) \left[\hat{c}_i(\omega) - \int \delta_{ij} d_j(\omega) d\hat{N}_{ij}\right] \\
& + \sum_{i \in I} \int \hat{v}_{ij} \left[dN_{ij} - d\hat{N}_{ij}\right] \geq \sum_{i \in I} \int \hat{v}_{ij} \left[dN_{ij} - d\hat{N}_{ij}\right],
\end{aligned}
$$

where the last inequality follows from the complementarity slackness for (c, N), and from the feasibility of $(\hat{c}, \hat{N})$. Now since both N and $\hat{N}$ are feasible, we have that:

$$
\hat{p} \cdot \bar{N} = \hat{p} \cdot \sum_{i \in I} N_{ij} = \hat{p} \cdot \sum_{i \in I} \hat{N}_{ij}.
$$

Hence, adding and subtracting $\hat{p} \cdot \bar{N}$, we obtain:

$$
\sum_{i \in I} \int \hat{v}_{ij} \left[dN_{ij} - d\hat{N}_{ij}\right] = \sum_{i \in I} \left[\hat{p} \cdot \hat{N}_{ij} - \int \hat{v}_{ij}\, d\hat{N}_{ij}\right] - \sum_{i \in I} \left[\hat{p} \cdot N_{ij} - \int \hat{v}_{ij}\, dN_{ij}\right] \geq 0
$$

where the last inequality follows from the first-order condition with respect to N.

B.5 Proof of Lemma 19

A solution to the agent's problem, (c_i, N_i), maximizes the Lagrangian:

$$
\begin{aligned}
L(\hat{c}_i, \hat{N}_i) = U_i(\hat{c}_i) \quad & + \quad \lambda_i \left[\bar{n}_i\, p \cdot \bar{N} + \sum_{\omega \in \Omega} q(\omega) \int d_j(\omega)\, d\hat{N}_{ij} - p \cdot N - \sum_{\omega \in \Omega} q(\omega) c_i(\omega)\right] \\
& + \quad \sum_{\omega \in \Omega} \mu_i(\omega) \left[\hat{c}_i(\omega) - \int \delta_{ij} d_j(\omega)\, d\hat{N}_{ij}\right],
\end{aligned}
$$

with respect to $(\hat{c}_i, \hat{N}_i) \in X_i$. This implies that the function $\beta \mapsto L(\beta c_i, \beta N_i)$ is maximized at $\beta = 1$. Taking first-order condition with respect to β at $\beta = 1$, and using the complementary slackness conditions, yields the desired result.

B.6 Proof of Proposition 20

Necessity. let (c, N, p, q) be an equilibrium. Since $\bar{n}_i > 0$, it follows from the firs-order conditions to the agent's problem that $\lambda_i > 0$. By direct comparison of first-order conditions, one can then verify that the equilibrium allocation solves the Planner's Problem with weights

$$\alpha_i = \frac{1/\lambda_i}{\sum_{k\in I} 1/\lambda_k}.$$

The associated Lagrange multipliers are $\hat{\mu}_i(\omega) = \alpha_i\mu_i(\omega)$, $\hat{q}(\omega) = \beta q(\omega)$ and $\hat{v}_{ij} = \beta v_{ij}$ and $\hat{p} = \beta p$, where $\beta \equiv \left[\sum_{i\in I} 1/\lambda_i\right]^{-1}$. Finally, we have from Lemma 19 that:

$$\alpha_i \sum_{\omega\in\Omega} \frac{\partial U_i}{\partial c_i(\omega)} c_i(\omega) + \bar{n}_i \hat{p} \cdot \bar{N}.$$

Adding up across all $i \in I$ and using $\sum_{i\in I} \bar{n}_i = 1$ yields the desired condition.

Sufficiency. Consider any solution of the Planner's problem satisfying the conditions stated in the Proposition. Notice that the second condition implies that $\alpha_i > 0$. Using Proposition 16 we obtain associated multipliers $\hat{q}$, $\hat{\mu}$ and $\hat{p}$. Consider then the candidate equilibrium prices $q(\omega) = \hat{q}(\omega)$ and $p = \hat{p}$. Then, by direct comparison of first-order conditions, one sees that the component (c_i, N_i) of the Planner's allocation solves agent $i \in I$'s problem, except perhaps for the budget feasibility condition. The associated multipliers are $\lambda_i = 1/\alpha_i$, $\mu_i(\omega) = \hat{\mu}_i(\omega)/\alpha_i$ and $v_{ij} = \hat{v}_{ij}$. To complete the proof, we thus need to verify that (c_i, N_i) satisfies budget balance:

$$\begin{aligned}
& \sum_{\omega\in\Omega} q(\omega)c_i(\omega) + p\cdot N_i - \bar{n}_i p \cdot \bar{N} - \sum_{\omega\in\Omega} q(\omega) \int d_j(\omega)\, dN_{ij} \\
= \quad & \sum_{\omega\in\Omega} \left[\alpha_i \frac{\partial U_i}{\partial c_i(\omega)} + \hat{\mu}_i(\omega)\right] c_i(\omega) + \hat{p}\cdot N_i - \bar{n}_i\hat{p}\cdot\bar{N} - \sum_{\omega\in\Omega} \hat{q}(\omega) \int d_j(\omega)\, dN_{ij} \\
= \quad & \sum_{\omega\in\Omega} \alpha_i \frac{\partial U_i}{\partial c_i(\omega)} c_i(\omega) - \bar{n}_i\hat{p}\cdot N + \int \left[\hat{p}_j - \hat{v}_{ij}\right] dN_{ij} \\
= \quad & \sum_{\omega\in\Omega} \alpha_i \frac{\partial U_i}{\partial c_i(\omega)} c_i(\omega) - \bar{n}_i\hat{p}\cdot N.
\end{aligned}$$

where we substituted in the Planner's first order conditions. But $\sum_{i\in I} \bar{n}_i = 1$ implies that:

$$\hat{p}\cdot N = \sum_{k\in I}\sum_{\omega\in\Omega} \alpha_k \frac{\partial U_k}{\partial c_k(\omega)} c_k(\omega),$$

hence budget balance holds since (c, N) satisfied the second condition stated in the Proposition.

B.7 Proof of Proposition 21

Proof that $\Delta^\star(\alpha)$ is convex-valued. To show that $\Delta^\star(\alpha)$ is convex valued, we note that when $u_i(c)$ is strictly concave, $c_i(\omega)$ is uniquely determined, and so the term

$$\pi(\omega)u_i'\left[c_i(\omega)\right] c_i(\omega)$$

is the same for all $(c, n) \in \Gamma^\star(\alpha)$. When $u_i(c)$ is linear, then $u'(c)c = c$ is linear. Taken together, this means that the function defining $\Delta^\star(\alpha)$ preserves the convexity of $\Gamma^\star(\alpha)$.

Proof that $\Delta^\star(\alpha)$ has a closed graph. Consider any converging sequence of α^k and $\Delta^k \in \Delta^\star(\alpha^k)$, generated by a sequence $(c^k, N^k) \in \Gamma^\star(\alpha^k)$. Since $\Gamma^\star(\alpha^k)$ is including in the set if incentive feasible allocation, which by Lemma 14 we know is weakly compact, we can extract a weakly convergent subsequence (c^ℓ, n^ℓ) of (c^k, n^k). Since we know from Proposition 15 that $\Gamma^\star(\alpha)$ has a weakly closed graph, it follows that $\lim(c^\ell, n^\ell) \in \Gamma^\star\left(\lim \alpha^\ell\right)$. If $u_i'(c)$ is continuously differentiable at $\lim c_i^\ell(\omega)$, then by continuity we have:

$$\lim \left(\alpha_i^\ell u_i'\left[c_i^\ell(\omega)\right] c_i^\ell(\omega)\right) = \left(\lim \alpha_i^\ell\right) \times u_i'\left[\lim c_i^\ell(\omega)\right] \times \left(\lim c_i^\ell(\omega)\right).$$

If $u_i(c)$ is not continuously differentiable at $\lim c_i^\ell(\omega)$ then given our maintained assumption that $u_i(c)$ is continuously differentiable over $(0,\infty)$, it must be that $\lim c_i^\ell(\omega) = 0$ and $u_i'(0) = +\infty$. Since $\lim c_i^\ell(\omega) = 0$ is part of a social optimum, it must be that $\lim \alpha_i^\ell = 0$. But we know in this case from Proposition 15 that

$$\lim \alpha_i^\ell u_i'\left[c_i^\ell(\omega)\right] c_i^\ell(\omega) = 0 = \lim \alpha_i^\ell u_i'\left[\lim c_i^\ell(\omega)\right] \lim c_i^\ell(\omega).$$

Taken together, we obtain that $\lim \Delta^\ell = \lim \Delta^k \in \Delta^\star(\lim \alpha^\ell) = \Delta^\star(\lim \alpha^k)$.

Proof that $\Delta^\star(\alpha)$ is bounded. Otherwise, there would exists some sequence α^k and $\Delta^k \in \Delta^\star(\alpha^k)$ such that $\max|\Delta_i^k| \to \infty$. Since α^k belongs to a compact set we can extract a converging subsequence α^ℓ. Since $\Delta^\star(\alpha)$ has a closed graph $\lim \Delta^\ell \in \Gamma^\star(\lim \alpha^\ell)$ and so must be finite, which is a contradiction.

An auxiliary fixed-point problem. Let M be such that $\max|\Delta_i| \leq M$ for all $\Delta \in \Delta^\star(\alpha)$ and $\alpha \in \mathcal{A}$. Let $\mathcal{D}$ be the set of transfers $\Delta = (\Delta_1, \dots, \Delta_I)$ such that $\sum_{i \in I} \Delta_i = 0$ and $\max|\Delta_i| \leq M$. Finally, let $K(\alpha, \Delta)$ be the function $\mathcal{A} \times \mathcal{D} \to \mathcal{A}$ such that

$$K_i(\alpha, \Delta) = \frac{(\alpha_i - \Delta_i)^+}{\sum_{k \in I}(\alpha_k - \Delta_k)^+},$$

where x^+ denotes the positive part of x. For each $(\alpha, \Delta) \in \mathcal{A} \times \mathcal{D}$, let the set $\Phi(\alpha, \Delta)$ be the product of the singleton $\{K(\alpha, \Delta)\}$ and the set $\Delta^\star(\alpha)$. By construction, $\Phi(\alpha, \Delta) \subseteq \mathcal{A} \times \mathcal{D}$. Since $\sum_{k \in I}(\alpha_k - \Delta_k)^+ \geq \sum_{k \in I}(\alpha_k - \Delta_k) = 1 > 0$ it follows that $K_i(\alpha, \Delta)$ is a continuous function over $\mathcal{A} \times \mathcal{D}$. Given our earlier result that $\Delta^\star(\alpha)$ has a closed graph, this implies that the correspondence $\Phi(\alpha, \Delta)$ has a closed graph as well. This allows to apply Kakutani's fixed point Theorem (see Corollary 17.55 in Aliprantis and Border (1999)) and assert that Φ has a fixed point, i.e., there exists some $(\alpha, \Delta) \in \mathcal{A} \times \mathcal{D}$ such that

$$\begin{aligned} \alpha_i &= \frac{(\alpha_i - \Delta_i)^+}{\sum_{k \in I}(\alpha_k - \Delta_k)^+} \text{ for all } i \in I \\ \Delta &\in \Delta^\star(\alpha). \end{aligned}$$

Proof that all fixed-points are such that $\Delta_i = 0$ for all $i \in I$. Next, we show that a fixed point of Φ has the property that $\Delta_i = 0$ for all $i \in I$. Indeed if $\alpha_i = 0$, then from the definition of $\Delta^\star(\alpha)$ we have that $\Delta_i \leq 0$, and from the fixed-point equation that $(-\Delta_i)^+ = 0 \Leftrightarrow \Delta_i \geq 0$. Hence, if $\alpha_i = 0$, then $\Delta_i = 0$. If $\alpha_i > 0$, then from the fixed point equation

$$\alpha_i \times \sum_{k \in I} (\alpha_k - \Delta_k)^+ = \alpha_i - \Delta_i \Rightarrow \Delta_i = \alpha_i \times \left[1 - \sum_{k \in I} (\alpha_k - \Delta_k)^+\right].$$

Hence, all Δ_i such that $\alpha_i > 0$ have the same sign. Since $\Delta_i = 0$ when $\alpha_i = 0$, it follows that all Δ_i have the same sign. But since $\sum_{i \in I} \Delta_i = 0$, this implies that $\Delta_i = 0$ for all $i \in I$.

B.7.1 Proof of Lemma 22

Consider two sets of weights α and α' with corresponding optimal allocations $(c, N) \in \Gamma^\star(\alpha)$ and $(c', N') \in \Gamma^\star(\alpha')$. Since the constraint set of the planner does not depend on α, (c, N) and (c', N') are both incentive feasible given α and α'. Hence, optimality implies that:

$$\alpha_1 U_1(c_1) + \alpha_2 U_2(c_2) \geq \alpha_1 U_1(c_1') + \alpha_2 U_2(c_2') \Leftrightarrow \alpha_1 \left[U_1(c_1) - U_1(c_1')\right] + \alpha_2 \left[U_2(c_2) - U_2(c_2')\right] \geq 0.$$

Vice versa:

$$\alpha_1' \left[U_1(c_1') - U_1(c_1)\right] + \alpha_2' \left[U_2(c_2') - U_2(c_2)\right] \geq 0.$$

Adding up these two inequality and using that, since the weight add up to one, $\alpha_1' - \alpha_1 = \alpha_2 - \alpha_2'$, we obtain:

$$\left[\alpha_1' - \alpha_1\right] \left\{\left[U_1(c_1') - U_1(c_1)\right] - \left[U_2(c_2') - U_2(c_2)\right]\right\},$$

which implies that:

$$U_1(c_1') - U_1(c_1) \geq U_2(c_2') - U_2(c_2).$$

But then we must have that

$$U_1(c_1') - U_1(c_1) \geq 0 \geq U_2(c_2') - U_2(c_2).$$

because otherwise either (c, N) or (c', N') would not be constrained Pareto optima.

B.7.2 Modified Security Market Line

Proposition 24 *Suppose the distribution of tree supplies is strictly increasing. Let $R_j(\omega) = \frac{d_j(\omega)}{p_j}$ be the return of asset j, $R_m(\omega) = \int_0^1 \frac{p_j}{\int_0^1 p_\ell \, d\bar{N}_\ell} R_j(\omega) d\bar{N}_j$ the market return, and $\beta_j = \frac{Cov(R_m, R_j)}{V(R_m)}$ the market beta of asset j. Then, β_j is a continuous and strictly decreasing function of j. Moreover, the expected return of tree j is a piecewise linear function of*

β_j:

$$\mathbb{E}[R_j - R_f] = \beta_j \Big(\mathbb{E}[R_m - R_f] - \theta_m \Big) + \theta_j, \tag{39}$$

where

$$\theta_j = \theta_k - \phi \max(\beta_j - \beta_k, 0\big) - \psi \max(\beta_k - \beta_j, 0\big), \tag{40}$$

and $R_f = \big(\sum_{\omega\in\Omega} q(\omega)\big)^{-1}$ is the risk-free rate, $\theta_j = \Delta_j/p_j$, is the (per dollar invested) divertibility discount of asset j, k is the marginal tree, $\phi > 0$, $\psi > 0$, and $\theta_m = \int_0^1 \frac{p_j}{\int_0^1 p_\ell d\bar{N}_\ell} \theta_j d\bar{N}_j$ is the average divertibility discount. Equation (39) also holds for financial assets by setting $\theta_j = 0$.

Proof that $j \mapsto \beta_j$ is strictly decreasing. Since there are only two states of nature, correlations are either equal to one, zero, or minus one. It follows from $R_m(\omega_1) < R_m(\omega_2)$ that $\beta_j = \frac{\sigma(R_j)}{\sigma(R_m)}\mathrm{Sign}[d_j(\omega_2) - d_j(\omega_1)]$, where:

$$\Big(\sigma(R_j)\Big)^2 = \sum_{\omega\in\Omega} \pi(\omega) \left(\frac{d_j(\omega) - \bar{d}_j}{p_j}\right)^2 = \sum_{\omega\in\Omega} \pi(\omega)(1-\pi(\omega))^2 \left(\frac{d_j(\omega_2) - d_j(\omega_1)}{p_j}\right)^2$$

Equation (**??**) implies that $p_j = a_i(\omega_1)d_j(\omega_1) + a_i(\omega_2)d_j(\omega_2)$, where i denotes the agent holding asset j and $a_i(\omega) > 0$. Thus:

$$\beta_j = \frac{1}{\sigma(R_m)} \left(\sum_{\omega\in\Omega} \pi(\omega)(1-\pi(\omega))^2\right)^{\frac{1}{2}} \frac{\frac{d_j(\omega_2)}{d_j(\omega_1)} - 1}{a_i(\omega_1) + a_i(\omega_2)\frac{d_j(\omega_2)}{d_j(\omega_1)}}. \tag{41}$$

$\frac{d_j(\omega_2)}{d_j(\omega_1)} \mapsto \beta_j$ is clearly continuous away from the marginal asset k. And it is also continuous at the marginal asset since p_j is continuous at $j = k$. For $j \neq k$, we can take the derivative:

$$\frac{d\beta_j}{d\frac{d_j(\omega_2)}{d_j(\omega_1)}} = \frac{1}{\sigma(R_m)} \left(\sum_{\omega\in\Omega} \pi(\omega)(1-\pi(\omega))^2\right)^{\frac{1}{2}} \frac{a_i(\omega_1) + a_i(\omega_2)}{\Big(a_i(\omega_1) + a_i(\omega_2)\frac{d_j(\omega_2)}{d_j(\omega_1)}\Big)^2} > 0.$$

Proof of equation (39). There is a different pricing kernel for each agent. For assets j held by agent i, the pricing kernel is:

$$1 = \mathbb{E}\left[\frac{q(\omega)}{\pi(\omega)} R_j(\omega)\right] - \delta \frac{\mu_i(\omega_i)}{\lambda_i} R_j(\omega_i).$$

Denoting the risk-free rate as $R_f = \big(E\big[\frac{q(\omega)}{\pi(\omega)}\big]\big)^{-1}$, the usual manipulations lead to:

$$\mathbb{E}[R_j(\omega) - R_f] = -R_f Cov\left(\frac{q(\omega)}{\pi(\omega)}, R_j(\omega)\right) + \theta_j,$$

where $\Delta_j = R_f \delta \frac{\mu_i(\omega_i)}{\lambda_i} R_j(\omega_i)$. Since there are two states of nature, $\frac{q(\omega)}{\pi(\omega)}$ can be written as an affine function of the market return with slope κ. Thus:

$$\mathbb{E}[R_j(\omega) - R_f] = -\kappa R_f Cov\big(R_m(\omega), R_j(\omega)\big) + \theta_j, \tag{42}$$

where $\theta_j = R_f \delta \frac{\mu_i(\omega_i)}{\lambda_i} R_j(\omega_i) = \frac{\Delta_j}{p_j}$. Multiplying by $\frac{p_j}{\int_0^1 p_\ell d\bar{N}_\ell}$ and integrating over j, we obtain the pricing kernel for the market portfolio:

$$\mathbb{E}[R_m(\omega) - R_f] = -\kappa R_f Var\big(R_m(\omega)\big) + \theta_m, \tag{43}$$

where $\Delta_m = \int_0^1 \frac{p_j}{\int_0^1 p_\ell d\bar{N}_\ell} \theta_j d\bar{N}_j$. Combining (42) and (42) yields the modified CAPM formula in Proposition **??**.

Next, we show that θ_j can be written as a piecewise linear function of β_j with a kink at the marginal asset β_k. $R_j(\omega_1) = \frac{d_j(\omega_1)}{p_j} = \frac{1}{a_i(\omega_1)+a_i(\omega_2)b_j}$, where i denotes the agent holding asset j and $b_j \equiv \frac{d_j(\omega_2)}{d_j(\omega_1)}$. Equation (41) implies that β_j can be written as a function of b_j: $\beta_j = \rho_0 \frac{b_j - 1}{a_i(\omega_1)+a_i(\omega_2)b_j}$, where $\rho_0 = \frac{1}{\sigma(R_m)} \left(\sum_{\omega \in \Omega} \pi(\omega)(1-\pi(\omega))^2\right)^{\frac{1}{2}}$. Inverting this function, we can write b_j as a function of β_j: $b_j = \frac{\rho_0 + \beta_j a_i(\omega_1)}{\rho_0 - \beta_j a_i(\omega_2)}$. Thus: $R_j(\omega_1) = \frac{\rho_0 - \beta_j a_i(\omega_2)}{(a_i(\omega_1)+a_i(\omega_2))\rho_0}$. Similarly: $R_j(\omega_2) = \frac{\rho_0 + \beta_j a_i(\omega_1)}{(a_i(\omega_1)+a_i(\omega_2))\rho_0}$. It implies that Δ_j is linear and decreasing in β_j for assets j held by agent 1 and linear and increasing for asset held by agent 2. It follows from the continuity of θ_j at the marginal asset k that θ_j can be written as (40).