Diversification and systemic risk

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Abstract

Portfolio diversification makes investors individually safer but creates connections between them through common asset holdings. Such connections create “endogenous covariances” between assets and investors, and enhance systemic risk by propagating shocks swiftly through the system. We provide a theoretical model in which shocks spread through constrained selling from N diversified portfolio investors in a network of asset holdings with home bias, and study the desirability of diversification by comparing the multivariate distribution of implied losses for every level of diversification. There may be a region on the parameter set for which the propagation effect dominates the individually safer one. We derive analytically the general element of the covariance between two assets $i$ and $j$, and discuss the factors on which it depends. Going further, we find agents may minimize their exposure to endogenous risk by spreading their wealth across more and more distant assets. The resulting network enhances systemic stability.

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*Keywords:* Systemic risk; Diversification; Circulant network; Circulant matrix, Fire sales

1 Introduction

When thinking about financial markets and systemic risk, one can find it useful to consider a group of climbers roped together at the top of a cliff. Each climber individually favors being roped as
it lowers his chances of falling off, yet one climber slipping now threatens the stability of his neighbors. The effect of being roped on the probability that many or all climbers fall is thus a priori ambiguous.

Prior to the 2008 credit crunch, both market participants and regulatory instances seemed to favor the roped equilibrium, implicitly assuming that individual soundness leads to systemic soundness. Yet the swiftness with which risk spread throughout the market, largely unanticipated, led to a shift to a more “connection-based” approach. Basel III will apply an extra capital ratio requirement of up to 2.5% to well connected establishments. Measures of systemic importance that account for externalities, such as Covar (Adrian and Brunnermeier, 2011), or Shapley values (Tarashev et al., 2010) have recently gained in popularity.

This change in focus has implications on the desirability of portfolio diversification from society’s perspective. Indeed while the individually risk reducing effect of diversification has been known since Markowitz (1952), diversification also forms “connections” between investors through common asset holdings, identified as a major carrier of contagion in the presence of fire sales (Shleifer and Vishny, 2012). The goal of this paper is to take a first step towards quantifying this contagion externality and comparing it to the individual risk-reducing effect, in order to get a primary assessment of diversification’s total impact on systemic risk, as well as the factors on which this impact may depend.

We do so by specifying and solving a theoretical model in which, in a given network, investors react to stochastic shocks on their wealth by selling/buying assets, which further impact asset prices and wealth, and so on. The model generates a normal multivariate distribution of investors wealth. Systemic risk is then studied through the probability that a large number of investors fall below a given bankruptcy threshold K, for different levels of diversification.

This approach allows us to map the individually risk reducing effect and the contagion externality into two distinct components of systemic risk: an individually safer investor implies a lower probability that one investor goes bankrupt, while a higher contagion externality means a higher likelihood that a large number X of investors fall, conditional on one bankruptcy. Therefore the
former affects the marginal distribution of each element on the vector of investors wealth, while the latter is contained within the dependence structure between all N investors. To obtain more clear-cut conclusions on the desirability of each level of diversification, we attach to each outcome a cost that grows exponentially with the number of failures, to capture the non-linear impact of financial losses on the real economy.

We find intermediate numbers of bankruptcies are less likely with a high degree of diversification, but the probability that many or all investors fail simultaneously is larger. In a context of high selling constraint and weak demand, the probability of “extreme failure” outcomes becomes non-trivial, so that no or little diversification may become optimal for society. We then introduce a touch of heuristics in the model by allowing agents on the demand side to grow more risk averse in a high variance environment. We find this strongly enhances the desirability of higher levels of diversification through a more indirect channel: spreading shocks across assets lowers the scope for panic. Ultimately, intermediate levels of diversification appear particularly harmful as they provide linkages through which shocks may spread without going far enough in minimizing individual risk and the likelihood that a panic occurs.

Our set-up also allows us to derive an analytical expression for the covariance between assets and investors $i$ and $j$ by diagonalizing the investor and asset covariance matrices. In our baseline network investors acquire assets which have low informational distance with those they already own, which we label as “home bias”. In this case covariances depend negatively on the distance between $i$ and $j$ in the financial network and on the number of assets in the economy. We show that an investor may minimize the variance of his portfolio by seeking assets which are far away from his in the financial network. We discuss the systemic implications of an “optimal” network in which all investors are no longer biased, and that of a “wider” network, ie one in which there are more assets. Moving to an optimal network is generally desirable for systemic risk, while moving to a wider one is unambiguously so. A mix of more optimality and width is particularly efficient, particularly for intermediate levels of diversification which previously were the most dangerous from a systemic perspective.
Previous work on the link between diversification and systemic has been split between between three strands of literature which have rarely crossed.

The first concerns small-scale\textsuperscript{1} models of financial contagion that highlight different channels through which common asset holding may lead to fire sales, which may in turn degenerate into systemic events. Schinasi and Smith (2000) for instance show that routine portfolio rebalancing brings contagion. The scope is increased when agents are subject to wealth effects as in Kyle and Xiong (2001). Goldstein and Pauzner (2004) point out that fire sales may also result from strategic risk\textsuperscript{2}.

The second strand of literature is based on statistical analysis. One method, taken by Shaffer (1994) or Wagner (2010), is to compare statistically a fully diversified situation to a fully undiversified one. Both authors show that the risk that all investors fail simultaneously is necessarily higher in the fully diversified situation. Our study confirms this fact but also goes more in depth by considering any level of diversification and any number of failures. A second approach deals with “fat tails”, which may mitigate the strength of the variance reducing effect, as showed by Samuelson (1967). In particular, Ibrahimov et al. (2011) use an indicator of the tail behavior of returns to define a diversification threshold. They find that on a given parameter range there can exist a wedge between investors interests and society ones.

The link between these first two strands is the correlation structure between asset returns. On one hand in any contagion model correlations between asset returns appear endogenously, as even two securities which are “fundamentally” independent become linked through the investors who hold them. On the other hand high correlations between assets lead to portfolios exhibiting “fat tails”. Asset correlations may then be seen as the output of the “contagion model” literature, and

\textsuperscript{1}The models usually feature only 2 assets, and diversification is defined as how evenly an investor spreads his wealth across both. A notable exception is provided by Lagunoff and Shreft (1999) who try moving to a 3 assets case. They find that the scope for contagion is decreased.

\textsuperscript{2}Others authors relate diversification to different amplification mechanisms, which deserve to be mentioned although they will not feature as such in this paper. Market distress may lead to an even wider collapse if it turns into liquidity distress: Adrian and Shin (2010) shows that leverage is negatively correlated to the market value of assets. Allen et al. (2010) find that when investors need to roll debt over, being connected brings a negative reputation externality. We also leave out a potential impact of diversification on “banks not doing their homework”. For instance, Jiao et al (2013) show that diversification may reduce the heterogeneity of investors beliefs, Calvo and Mendoza (2000) argue that diversification lowers the incentive for investors to acquire information about securities before selling.
Finally the third strand of literature deals with network analysis of financial stability. In particular, this paper may be related to previous work on the “robust yet fragile” feature of the financial system, as in Nier et al. (2007) or Amini et al (2012).

The broad method of this paper is to bridge these three approaches, by proceeding in three steps:

1) setting up a large scale contagion model. \(N\) constrained “portfolio investors” who hold from 1 to \(N\) assets are forced to sell to “convergence traders” in response to negative shocks on their wealth. This lowers prices, further tightening the constraint, and so on. Mathematically this translates into a linear system of \(N\) recurrence equations of price returns, in which the strength of the recurrence will depend on the magnitude of the constraint and the discount on the sales.

2) specifying a network of asset holdings. We set two characteristics for investors: they are home biased, and all have the same pattern of asset holdings. This leads respectively to a financial network in which neighbors share relatively more connections, and is circulant. The matrix of asset holdings is circulant also which allow us to solve analytically the system, obtaining well-behaved \(N\) by \(N\) covariance matrices between assets and investors\(^3\).

3) studying systemic risk through statistical analysis and analytical expression of the covariances. Statistical analysis is run through the multivariate distribution of portfolio losses, which gives us the likelihood that any number of investors between 0 and \(N\) fail, for a given level of diversification \(n\).

To the best of our knowledge two recent papers have used a similar approach in terms of obtaining endogenous covariances from a theoretical model. Danielsson, Shin, and Zigrand (2011) define a multivariate model which produces a matrix of covariance of unrestricted dimensions. The correlations obtained have a “fundamental” and an “endogenous risk” component, where endogenous risk is the risk resulting from “the actions of market participants which are hard wired in the system”. Cont and Wagalath (2012) specify a similar but more aggregated model, which they cal-

\(^3\)Circulant matrices are are a useful tool, and a side goal of the paper is to contribute to widening their use and understanding beyond the fields of pure network analysis and signal theory in which they are more common.
ibrate to estimate the realized covariance matrix during well-known fire sales episodes, such as the aftermath of the collapse of Lehman brothers.

Our study places itself within this endogenous risk approach, but differs from these papers in that it puts the network of asset holding at the center of the analysis. In spirit, both papers focus on explaining the pattern of prices correlation during crisis episodes, while our interest lies primarily with the desirability of diversification from a systemic perspective.

Finally, the paper shares some of its conclusions with Caccioli et al. (2012) who find that there might be a window on the diversification spectrum for which systemic risk may become significant. Caccioli et al. study the asymptotic properties of the financial system as a function of the average number of connections of a node in an otherwise random network. Our model is micro-founded, and specifies a given network based on economic evidence. This gives the model a greater granularity, as we show how different micro foundations lead to different conclusions on systemic risk, even within a network whose general characteristics remain unchanged.

In short, the model is somewhat “hybrid” between micro-founded and network approaches, which allows it to remain both tractable and general, and provide a wide oversight of the factors which may impact on the link between diversification and systemic risk.

Section 2 presents the baseline model, and how it brings recurrence in asset prices. Section 3 presents the distributions of investor wealth for the baseline model, and that with possible panics. In section 4 we go further by drawing and discussing the realized covariances between assets and investors, and studying the impact of a wider and/or a home bias free network on systemic risk. Section 6 concludes.

2 Set-up

The model presented below is designed to generate a linear system which will allow us to draw N-dimensional covariance matrices between assets. Vectors are indicated with bold characters.
2.1 The market

The investment period starts at \( t=0 \) and finishes at \( t=T \), where each period \( t \mapsto t+1 \) may be seen as “a day” on the markets, and \( T \) is a large but finite number. Financial markets are composed of \( N \) risky assets, which are best seen as stocks. \( N \) is finite to keep the individual benefits from diversification within bounds, so that markets are incomplete. Asset prices have three sources of movement:

\[
\Delta P_t = \Delta P^F_t + MR_{t+1} + e^*_t
\]

where \( \Delta P_t \) is the *actual price* evolution between \( t \) and \( t+1 \), \( \Delta P^F_t \) the *fundamental price* evolution, \( MR_{t+1} \) is the *mean reversion* vector, and \( e^*_t \) is the *investor-driven deviations* from fundamental value. All elements are \( N \times 1 \) vectors. Intuitively, these dynamics imply that the behavior of investors may induce actual prices, which are defined in equilibrium, to deviate from their fundamental value by an amount \( e^*_t \). Such deviations however tend to die out in the medium/long run, through \( MR_{t+1} \).

The *fundamental price/value* of each stock is defined as the discounted value of its future dividends. We assume dividends \( D \) follow an arithmetic Brownian motion, ie for stock \( i \) at time \( t \):

\[
D_{i,t+1} = D_{i,t} + u^D_t + e^D_{i,t+1}
\]

The Gordon growth model yields:

\[
p^F_{i,t} = \frac{D_{i,t}}{k} + cst,
\]

where \( k \) is an exogenous discount factor suited to the riskiness of the cash-flows, and \( cst \) a constant. Consequently \( p^F_{i,t} \) also follows an arithmetic Brownian motion, which we note:

\[
p^F_{i,t+1} = p^F_{i,t} + u^F_t + e^F_{i,t+1}
\]

where \( u^F_t \) is the *fundamental drift*, and \( e^F_{i,t+1} \) a normally distributed shock, \( N \sim (0, \sigma^2_F) \). Both are linear functions of the underlying dividend drift and shock: \( u^F_t = \frac{u^D_t}{k} \) and \( e^F_{i,t+1} = \frac{e^D_{i,t+1}}{k} \).

We assume that the fundamental drift is the same for all assets, and that fundamental shocks are independent across asset and time, ie \( u^F_t = u \) and vector \( e^F_t \), the \( N \times 1 \) vector of general element \( e^F_{i,t} \), \( N \sim (0, \Sigma_F = \text{diag} (\sigma^2_F)) \). These assumptions will simplify presentation and calculus, but also show how correlations between assets may arise from the actions of market participants, even starting
from a situation in which they are fundamentally independent. The vector of fundamental prices writes:

\[ \Delta P^F_t = u + e^F_{t+1} \]

The mean reversion component ensures that prices do not go too far away from their fundamental value. In our model mean reversion will be driven by an exogenous supply of “new” investors at each period \( t \), who will demand asset \( i \) if \( p_{i,t} - p^F_{i,t} < 0 \), and short sell it if \( p_{i,t} - p^F_{i,t} > 0 \). We set this amount of “new” investors to be quite low, so that prices only converge to their fundamental value in the medium/long run. Mathematically:

\[ \text{MR}_{t+1} = -\lambda (P_t - P^F_t) \]

where \( \lambda \), the speed of mean reversion, is fairly low.

Substituting for \( \Delta P^F_t \) and \( \text{MR}_{t+1} \) we may then rewrite the price vector as:

\[ \Delta P_{t+1} = u - \lambda (P_t - P^F_t) + e^F_{t+1} + e^*_t + e^*_t + e^*_t + e^*_t \]

\[ (1) \]

\( e^*_t \), the investor driven deviations vector, will be the focal point of this paper. It gives the movement in prices resulting from the actions/constraints of investors that may cause prices to deviate from their fundamental value. Its exact dynamics will be derived in the following section, but it is useful to give the outline in words.

\( e^*_t \) is derived from the interactions of two types of agents: “portfolio investors”, whose strategy is based on fundamental drifts, and “convergence traders” who exploit the short-term deviations from fundamental value. We may also refer to them as long term and mid term investors due to their investment horizon, which we note respectively LT and MT. LT investors face a constraint which forces them to sell in response to a negative shocks on their portfolio. MT investors buy those asset at a discount, which leads prices to fall below their fundamental value. This brings LT investors to sell further, and so on.

The full specialization between long-term and mid-term investors results from an ex-ante arbitrage by both agents. Convergence traders specialize on spotting and exploiting deviations of prices
from their fundamental value, while portfolio investors have a more passive approach. Formally, we assume MT investors pay a fixed cost $\epsilon_1$ to be able to successfully measure $P_t - P^F_t$, while LT investors pay $\epsilon_2$ to correctly estimate the unconditional moments of vector $\Delta P_{t+1}$. Such a market segmentation is close to that of Graham who differentiates an “active or enterprising approach to investing” from a “passive or defensive strategy that takes little time or effort but requires an almost ascetic detachment from the alluring hullabaloo of the market” (Graham and Zweig, 2003, p101). One may also view LT investors as mutual funds or banks, while MT investors are more speculative agents following arbitrage strategies, such as hedges funds.

Long-term investors hold a diversified portfolio, with $n \in [1, N]$ the level of diversification. The level of diversification thus impacts the system in two ways. First, the total shock on the portfolio is likely to be lower when $n$ is large, so that the wealth shocks should be lower. Second, a given shock will trigger sales on many assets when $n$ is large, so that LT investors and assets become more correlated. Initially each long-term investor $I$ is endowed with “his” asset $i$, ie investor 1 holds asset 1, so that they are $N$ representative long term investors.

Note that in what follows we nearly only refer to constrained selling by portfolio investors, because we are primarily interested in fire sales. However the model applies equivalently to a situation in which portfolio investors are buyers, as their constraint gets looser following a positive shock on their wealth\footnote{More on this assumption in section 2.4}. Convergence traders are segmented as in Merton (1987).

We introduce some notation: for a given investor $I$, $q_{i,I}$ represents the actual quantity of asset $i$, $q_{i,I}^*$ the desired one, $q_I = \sum_{i=1}^N q_i$ is his total investment in all risky assets. $q_{i} = \sum_{I=1}^I q_{i,I}$ is the total quantity of asset $i$ across investors.

### 2.2 Investors

Both mid-term and long-term investors have CARA utility, with risk aversion of $\frac{1}{\tau_{mt}}$ and $\frac{1}{\tau_{lt}}$ respectively. Risk aversion is similar across investors of the same type, and each agent is a price-taker.
2.2.1 Mid-term investors

They hold assets during \( t^* \) periods, the time it takes for investor-driven deviations shocks to return to 0. They pay a fixed cost \( \varepsilon_1 \) to monitor those deviations at every \( t \). They are free of regulation and have “deep pockets”, so that they may hold exactly the quantities they desire: \( q^*_{i,t} = q_{i,t} \). As they are segmented each investor operates in one market only, so that we drop the vector notation for the time being. Mathematically each \( I \) solves:

\[
\max E\left(-e^{-\frac{w_{I,t+1^*}}{\delta_m}}\right)
\]

\[
u/c \ w_{I,t+1^*} = w_{I,t} + q^*_{i,t} (p_{i,t+1^*} - p_{i,t}) - t^* \varepsilon_1
\]

where \( w_{I,t} \) is the wealth of investor \( I \) at time \( t \). Using the moment generating function yields the well-known solution:

\[
q^m_{i,t} = \tau_m \frac{E(p_{i,t+1^*} - p_{i,t})}{E(\sigma^2_{t+1^*})}
\]

where \( \sigma^2_{t+1^*} \) is the variance of asset price \( i \) at horizon \( t + t^* \).

When \( t^* \) is large, all trading shocks, whether they have occurred or are expected to, will vanish through mean-reversion. Setting a fairly large value for \( t^* \) we thus have:

\[
E(p_{i,t+1^*} - p_{i,t}) = E(\sum_{j=t}^{j=t+1^*} \triangle p_{i,j}) = E(u^* + \sum_{j=t+1^*}^{j=t+1^*} MR_{i,j} + \sum_{j=t+1}^{j=t+1} e_{i,j}^* + \sum_{j=t+1}^{j=t+1^*} e_{i,j}^*)
\]

\[
E(p_{i,t+1^*} - p_{i,t}) = t^* E(u) + E(p_{i,t+1}^F - p_{i,t})
\]

As mentioned, convergence traders successfully monitor deviations from fundamentals, so that

\[
E(p_{i,t+1}^F - p_{i,t}) = p_{i,t+1}^F - p_{i,t}.
\]

First differencing we obtain the demand/supply shock for a given convergence trader between \( t \) and \( t+1 \):

\[
\triangle q^m_{i,t} = \tau_m \frac{(p_{i,t+1}^F - p_{i,t+1}) - (p_{i,t}^F - p_{i,t})}{E(\sigma^2_{t+1^*})}
\]

Rearranging and setting to \( m \) the number of convergence traders operating on each market, we obtain the vector of demand/supply shifts from convergence traders between \( t \) and \( t+1 \):
\[\triangle Q_{t}^{mt} = -h(\triangle P_{t} - \triangle P_{t}^{F}) \quad (2)\]

where \(h = \frac{m\tau_{mt}}{E(\sigma_{t,r}^{2})}\) represents the total increase in the quantity demanded in response to a rise in expected returns, and thus indicates the strength of the demand. If the distance between actual and fundamental prices remains constant between \(t\) and \(t+I\), the demand by MT investors will be null. If \(\triangle P_{t} > \triangle P_{t}^{F}\) they will sell, if \(\triangle P_{t} < \triangle P_{t}^{F}\) they will buy.

Note also that starting from a situation in which actual prices are equal to fundamental prices, so that is no mean reversion, and portfolio investors are not constrained and thus do not supply/demand any assets, market equilibrium implies \(-h(\triangle P_{t} - \triangle P_{t}^{F}) = 0 \iff \triangle P_{t} = \triangle P_{t}^{F}\). Therefore in a “business as usual” scenario in which the constraint of portfolio investors does not bind, actual prices should be equal to fundamental ones at any time \(t\).

2.2.2 Long-term investors

Maximization problem

Their time horizon is noted \(t^{\circ}\). Formally:

\[
\max E\left(-e^{-\frac{w_{t,t+t^{\circ}}}{\lambda}}\right)
\]

\[
u/c \ w_{t,t+t^{\circ}} = w_{t,t} + Q_{t}^{T}(P_{t+t^{\circ}} - P_{t}) - t^{\circ}\varepsilon_{2}
\]

\[
E(P_{t+t^{\circ}} - P_{t}) = E(P_{t+t^{\circ}} - P_{t}) + \sum_{j=t+1}^{t+t^{\circ}} MR_{j} + \sum_{j=t+1}^{t+t^{\circ}} e_{t+1}^{F} + \sum_{j=t+1}^{t+t^{\circ}} e_{t+1} - P_{t})
\]

\[
= E(t^{\circ}u) + \lambda E(\sum_{j=t+1}^{t+t^{\circ}} (P_{j}^{F} - P_{j}))
\]

as \(E(e_{t+1}^{F}) = 0\), since LT investors do not pay the cost \(\varepsilon_{1}\) and do not observe the deviations of prices from their fundamental value. However as they incur the cost \(\varepsilon_{2}\) they successfully estimate the mean and covariance of vector \(P_{t+t^{\circ}} - P_{t}\), noted \(\Sigma_{t^{\circ}}\). Formally this means \(E(P_{t+t^{\circ}} - P_{t}) = t^{\circ}u\) and \(E(\Sigma_{t^{\circ}}) = \Sigma_{t^{\circ}}\). Therefore:

\[
Q_{t}^{lt} = \frac{t^{\circ}u / \Sigma_{t^{\circ}}}{\Sigma_{t^{\circ}}}
\]
Two key elements:
- Since the unconditional moments of $P_{t+1|t} - P_t$ are constant, their desired quantities will be constant.
- As we shall verify later in equilibrium, the matrix $\Sigma_{t|t}$ is symmetric. This implies that LT investors optimally want to hold an equal share of all assets, so that:

$$\frac{q_{I,t}}{q_{I,t}} = \frac{1}{n} \quad (3)$$

Note that optimally portfolio investors want to hold as many assets as possible to maximize the benefits of diversification. The level of diversification $n$ is thus equal to the maximum number of markets they have access to.

**constraint**

We assume a linear correspondence between the total amount of risky assets that an investor may hold at $t+1$ and his value-at-risk at $t$.

$$q_{I,t+1} \leq rVaR_{I,t}$$

where parameter $r$ mitigates the strength of the selling in response to a change in the VaR. In words, when the constraint binds, a change in the VaR at time $t$ will lead investors to sell/buy risky positions between $t$ and $t+1$ to comply with the constraint at $t+1$. Note that a more usual form would be $P_t Q_t \leq rVaR_{I,t}$, ie the VaR leads investors to manage their monetary exposures rather than their quantities solely\(^5\).

Each investor’s VaR is given by:

$$VaR_{I,t} = E_t(\mathbf{P}_{t+1|t})^\top \mathbf{G}_{I,t}^{lt} + cst \sqrt{(\mathbf{G}_{I,t}^{lt})^\top \Sigma_{t|t} \mathbf{G}_{I,t}^{lt}}$$

where $\mathbf{G}_{I,t}^{lt}$ is the vector summarizing the proportions of each asset in total quantity invested by investor $I$. Using (3) and the fact that $E(\mathbf{P}_{t+1|t}) = \mathbf{P}_t + t^\alpha \mathbf{u}$ for LT investors, we have $E_t(\mathbf{P}_{t+1|t})^\top \mathbf{G}_{I,t}^{lt} = \ldots$

\(^5\)more on this choice in section 2.4
\[ \frac{\alpha}{n} \left[ \left( \sum_i p_{i,I,I} + \tau^u \right)/n \right], \] where \( \sum_i p_{i,I,I} \) represents the sum of the prices of assets that feature in \( I \)'s portfolio. First differencing, we obtain the selling of a given asset \( i \) by investor \( I \) that operates on his constraint:

\[ \implies \Delta q_{i,I,I}^t = \frac{r}{n} \sum_i \Delta p_{i,I,I-1} \]  

(4)

The message conveyed by this equation is simple: when prices increase, investors are more wealthy and thus may invest more at the following period, when prices fall they must lower their exposure to risky assets. As long as the constraint does not bind the supply/demand by portfolio investors will be null, and prices will simply track fundamental values. The rest of the paper studies a situation in which the constraint binds for all LT investors.

Note that VaR constraints as such usually concern banks, so that one may argue the model applies primarily to markets dominated by banks, such as ABS. Yet trust funds may also be subject to an explicit constraint by their beneficiaries. What’s more, in practice fire sales may also be driven by increased risk aversion, higher emphasis on the short run, etc. The constraint chosen is only a tractable vehicle for such sales, but captures a wider phenomenon. Thus we avoid referring to LT investors as “banks” only, and believe our results apply to markets in general.

2.3 Network formation and matrix form

Network

The previous section showed how each representative investor \( I \) behaves. To study the implications of such behavior on each price we must specify a pattern of asset holdings. Figure 1 summarizes the network formation: each holding of asset \( i \) by investor \( J \) is a connection between \( J \) and the investor \( I \) who initially held the asset. Investors are nodes and asset holdings are the connections between them. The numbers on each link thus represent the assets that \( I \) and \( J \) have in common, indicating how closely related they are.

As the degree of diversification \( n \) increases, each \( I \) acquires the asset that is the closest to his
right. Then $I_k$ holds only asset $k$ if $n = 1$, assets $k$ and $k+1$ if $n = 2$, etc. In general $I_k$ holds assets $k$ to $k'$, where $k' \equiv k + n - 1$ [N]

The same information may be expressed in matrix form by letting each row represent an investor $I$ and each column an asset $i$, and setting $1$ if $I$ holds $i$, 0 otherwise. For instance if $N=5$ and $n=3$:

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$I_1$ holds assets $1,2,3$ so that $a_{1,1} = a_{1,2} = a_{1,3} = 1$.

This network has two important features:

- As each investor chooses to purchase the asset of his nearest “neighbor” in the financial network, he exhibits some degree of home bias. In the literature on international diversification, home bias refers to the well documented tendency of investors to favor assets with the lowest distance to those they already own. “Distance” between two assets $i$ and $j$ may reflect informational asymmetries between them, the transaction cost of acquiring one when possessing the other, the fact the investors who hold $i$ are not familiar with $j$, etc. Here we take no stance on what factor underlie “distance”, seeing it as a combination of heuristic and rational factors.

- All investors have a similar pattern of asset holdings, so that the network is circulant, and so is matrix $A$. Visually, in circulant matrices each row is equal to the previous row, shifted one spot.
to the right. It is important to stress that circulant networks are only the result of our assumption of symmetry across investors, not from the home bias feature mentioned. For instance a network in which investor 1 holds assets 1, 4, 5, is circulant if investor 2 holds 2,5,6, and so on. As we shall see in section 4, circulant matrices have powerful properties, which will allow us to diagonalize them easily to solve the model and provide analytical expressions for the covariance between asset \( i \) and \( j \).

**Matrix form**

From section 2.2.2 we know that the proportion of asset \( i \) in total quantity is \( 1/n \) if \( i \) features in the portfolio, and 0 otherwise. The average return on investor \( I \)'s portfolio writes:

\[
\frac{\sum_i a_{I,i} \Delta p_{i,t}}{n}
\]

And the selling constraint (4) may be re-expressed as:

\[
\Delta q_{i,I,t} = \frac{r}{n^2} \sum_i a_{I,i} \Delta p_{i,t-1}
\]

ie the amount of each asset sold by investor \( I \) is a linear function of the product of each line \( I \) of \( A \) by the price vector.

The vector of general element \( \Delta q_{i,I,t} \) summarizing the unit selling by each long-term investor \( I \) as a response to a wealth shock, may then be expressed as:

\[
\Delta q^I_t = \frac{r}{n^2} A \Delta P_{t-1}
\]

What’s more, just as each row of \( A \) tells us what assets a given investor holds, each row of \( A^\top \) indicates what investors hold a given asset \( i \). Thus we may express \( \Delta Q^I_t \), the vector whose general element \( \Delta q_{i,t} \) is the total selling of asset \( i \) between \( t+1 \) and \( t \), as:

\[
\Delta Q^I_t = A^\top \Delta q^I_t
\]

\[
\Rightarrow \Delta Q^I_t = \frac{r}{n^2} A^\top A \Delta P_{t-1}
\]

(5)
Let us now take the market clearing condition \( \Delta Q^l_t + \Delta Q^m_t + \Delta Q^mr_t = 0 \), where the terms on the right-hand side are the demand/supply shocks stemming from respectively portfolio investors, convergence traders, and the mean reverting exogenous supply of investors. Substituting:

\[
-h(\Delta P_t - \Delta P^F_t) + \Delta Q^mr_t + \frac{r}{n^2}A^TA\Delta P_{t-1} = 0
\]

\[
\Rightarrow \Delta P_t = \mathbf{u} + \frac{\Delta Q^mr_t}{h} + (\frac{r}{h n^2})A^TA\Delta P_{t-1} + \mathbf{e}^F_{t+1} \tag{6}
\]

The end product of the model is thus a stochastic recurrence system that maps how price shocks spread through time and assets. The asset price dynamics we fed into LT and MT investors maximization problems are verified in equilibrium. Most importantly we obtain the expression for the vector of investor-driven deviation from fundamentals: \( \mathbf{e}^*_{t+1} = (\frac{r}{h n^2})A^TA\Delta P_{t-1} \)

Summary of the frame of the model:
- portfolio investor of type I hold assets \( i=1 \) to \( i=I+n \)
  - when faced with a negative shock I sells an equal quantity of every asset, where the amount sold depends on \( r \), “the strength” of the constraint
  - convergence traders buy these assets with a discount that depends on \( h \), the “strength” of demand
  - The market clearing condition then yields \( \Delta P_t = \mathbf{u} + \frac{\Delta Q^mr_t}{h} + (\frac{r}{h n^2})A^TA\Delta P_{t-1} + \mathbf{e}^F_{t+1} \)

Figure 2 gives a chronological description of how a vector of shock at \( t=0 \) propagates through the system:

### 2.4 Assumptions

Before solving the model, it is appropriate to discuss 4 assumptions underlying it:

1) **Stock prices are normally distributed.** The relevance of this postulate has already been discussed extensively (Andersen et al., 2001): while admittedly a poor description of reality, its merit
Figure 2: sequence of events following a shock

is to be easy to manipulate, and also in our case to allow for the study of the multidimensional density function. We have chosen normality over log-normality, as the economic logic of the model dictates we work with price evolution rather than returns.

Nonetheless, one of the most common criticism attached to normal price returns is that they are incapable of capturing the actual thickness of the tails. Our model shows that under a particular set of circumstances the system endogenously generates a dependence structure between assets which make such “fat tails” appear. The paper thus avoids this pitfall.

2) The model is symmetric. This means we assume that negative shocks bring sales but equivalently positive shocks bring purchases. Symmetry comes partly from normality, but it also carries the implicit assumption that constrained investors may not escape the constrained region even when their wealth increases, that is the desired quantity of assets is always below that allowed by the constraint. In reality we may expect some investors not to buy assets in response to a looser constraint if they are happy with their holdings.

This assumption is purely to simplify analysis. One could include asymmetry in the model fairly easily, for instance by specifying a proportion of investors who escape the constraint binding
region for each positive price shift. Nonetheless, since we are only interested in what happen in the left-tail, the author has chosen not to burden the model with parameters that affect the right-tail of the distribution.

3) **Parameters are constant.** So far, agents stick with their assessment of asset moments, and never question this assessment. In particular, convergence traders always provide demand with the same discount $h$, regardless of the current market conditions. In practice increased risk aversion, or higher emphasis on the short-term may hamper demand during crises. We account for this in section 3 by allowing risk aversion to rise in response to high selling movements.

4) **The constraint relates linearly quantities and VaR.** As stated in section 2.2.2, a more common form relates monetary holdings and VaR. This form also has the advantage of being driven by theory as it is derived from the requirement that capital should exceed the VaR. However, as mentioned such a form does not allow to derive a market equilibrium, and we are forced to relate solely quantity to the VaR.

Though this choice clearly comes from necessity, it may be more justifiable on normative grounds. In reality the link between shocks on portfolios and selling decisions depends upon many factors, and its exact form is hard to know, so that a linear relationship may not be ruled out. For instance looking at the charts provided by Adrian and Shin (2010), a linear form between leverage and asset returns during crisis appears to be an acceptable fit.

### 3 Multivariate distribution of investor wealth

In this section we discuss the impact of diversification on systemic risk by studying the likely number of bankruptcies through the distribution of the vector of total change investor wealth. We then specify an objective function for society in which the cost is an exponential function of the number of failures.
3.1 Baseline model

3.1.1 Expected value and variance of total change in investor wealth

To convey maximum intuition while minimizing quantitative heavy lifting, we will study how a single stochastic shock spreads through time. We thus set all shocks to 0 past $t=0$ for all securities, letting total losses depend on the realization of the stochastic initial price fall.

We also set the mean-reversion and fundamental drift to 0, for three reasons: a) our time unit is a “day” in the market, so that the fundamental long-term trends should be negligible anyway, b) they would overload the equations as a recurrence system with a drift is slightly more demanding and c) without changing the insights of our model, since the variance structure would not be changed by the inclusion of $MR$ and $u$.

These simplifying assumptions lead to the following form for asset prices:

$$\triangle P_0 = e^F_1$$

$$\triangle P_t = \left(\frac{r/h}{n^2}\right)A^TA\triangle P_{t-1} \quad (7)$$

$$\Rightarrow \triangle P_t = \left[\left(\frac{r/h}{n^2}\right)A^TA\right]^t \triangle e^F_1 \quad (8)$$

which notably implies $E(\triangle P_t) = \left[\left(\frac{r/h}{n^2}\right)A^TA\right]^t E(e^F_1) = 0$.

We label as “change in the wealth of investor $i$” the evolution of the average price of the assets in $i$’s portfolio throughout the period. Since each row $i$ of matrix $A$ represents the holdings of investor $I$, this average price shift at period $t$ is $I_t = \frac{1}{n}A\triangle P_t$, and the total price return over the period, on which we shall focus, is:

$$I = \frac{1}{n}A \sum_{t=0}^{\infty} (\triangle P_t) = \frac{1}{n}A \sum_{t=0}^{\infty} \left[\left(\frac{r/h}{n^2}\right)A^TA\right]^t \triangle e^F_1$$

Its expected value is null, and its variance is given by: $\Sigma_I = E((I - E(I))(I - E(I))^\top)$, which yields:
\[
\Sigma_I = \frac{\sigma_F^2}{n^2} \left[ A \sum_{t=0}^{t=+\infty} \left( \frac{r/h}{n^2} \right) A^\top A \right]^\top \left[ A \sum_{t=0}^{t=+\infty} \left( \frac{r/h}{n^2} \right) A^\top A \right] \]

since \( E(\triangle \epsilon_1^F \triangle \epsilon_1^F^\top) = \text{diag}(\sigma_F^2) \) as the fundamental shocks are independent across assets. Two comments are of order before proceeding:

- \( I \) is a linear combination of the initial normally distributed shocks, so it is normally distributed.
- the system is not explosive if the eigenvalues of \( \frac{r/h}{n^2} A^\top A \) are below one. We later prove that the maximum eigenvalue of \( A^\top A \) is \( n^2 \). Therefore the convergence condition for \( \frac{r/h}{n^2} A^\top A \) is 
\[
\frac{r/h}{n^2} n^2 < 1,
\]
that is:
\[
r < h
\]

The increase in demand following a drop in prices must be superior to that of constrained supply, otherwise prices are ever-falling.

### 3.1.2 Choice of parameters

Results will depend on 4 parameters: the fundamental variance \( \sigma_F^2 \), the maximum loss \( K \) that investors can incur before bankruptcy, the number of assets \( N \), and finally the conditions on the markets \( r/h \). We choose our baseline parameter set to fit the state of the financial markets coming in to the 2008 credit crunch. We normalize prices and initial quantities at \( t=0 \) to 1.

To estimate the fundamental variance we compute the average daily variance of an asset belonging to the S&P 500, from the first of July 1997 to the first of July 2007. This yields \( \sigma_F^2 = 0.000616 \). The S&P was chosen because it is designed to provide a broader description of the investment opportunities than its counterparts, and equities are the asset class which suits our model the most. With respect to the period chosen, in our model noise is only fundamental in BAU so that the window does not include the subprime crisis. Nevertheless we include the internet bubble, which we considered as an increased volatility episode rather than a systemic event.
An investor goes bankrupt when his losses exceed his capital $K$. We use data of the World Bank to set $K=0.08$, that is 8% of normalized assets. This value stands between the reported capital ratios of Tier 1 capital for US banks of 9.1% in 2008, and that of European ones, usually around 5%. We have somewhat arbitrarily chosen to be on the high side, to reflect the higher weight of the US markets.

The first two parameters have a similar and straightforward impact in our model: a fundamentally riskier portfolio or a lower default threshold both make each investor marginally riskier. Thus for conciseness we keep parameters $\sigma_F^2$ and $K$ constant throughout the paper, relegating simulations with alternative values to Appendix A.

$N$ represents the number of assets in our model but in essence describes more the number of asset classes. Anecdotal evidence from investment funds implies a number of asset between 5 and 15. Furthermore, seeing $N$ as the maximum diversification level, we may follow a classic paper from Evans and Archer (1968), who estimate that diversification is no longer profitable past 10 securities, a belief shared amongst practitioners. We thus set $N=10$, for now. We will study in section 4 the impact of rising $N$ to 20, as in Cont and Wagalath (2012).

With respect to market conditions $r/h$, we study 3 scenarios:

- a “mild” one in which the constrained agents are forced to sell assets in fairly small quantities when a shock hits, and the demand by active investors for such assets is strong. $r/h = 0.6$
- a “windy” scenario in which the quantities sold by constrained agents and the demand by convergence traders is are both moderate. $r/h = 0.75$
- a “storm” scenario in which the pro-cyclical effect of the VaR constraint is large, and demand response is weak. $r/h = 0.9$

Studying 3 scenarios results partly from the lack of data on fire sales by investors, but is mostly interesting as part of our analysis.

---

6There has been empirical evidence on the presence of fire sales (see for instance Coval and Stafford (2007)), but it is hard to use it for calibration as researchers can only conjecture that a given sale has been made out of necessity, and their impact on prices.
3.1.3 Distribution of total number of bankruptcy

Figure 3 summarizes our approach of systemic risk in the intermediate scenario $r/h = 0.75$. Each possible number of investor failures, from 0 to $N=10$, has a probability given by the multivariate cumulative normal distribution, and we study how this probability varies with the level of diversification. The dual impact of diversification appears clearly. As $n$ rises investors become more and more dependent, outcomes in which some investors fail but other survive become less likely. In the total diversification case, only the “all survive” and the “all fail” outcomes are possible. Due to the individually risk-reducing impact of diversification, the “all survive” equilibrium increases faster in likelihood, yet we also observe a gradual detachment from 0 of the likelihood that every investor fail, reaching a non-trivial 0.005% for $n=10$. In this case the overall desirability is thus ambiguous, and will crucially depend on how painful mass failure is to the entire economy.

Figures 4,5,6 zoom on number of failures large enough to constitute a systemic event, for all values of with $r/h$. 

Figure 3: distribution of number of bankruptcy in stable regime with $r/h=0.75$
Figure 4: extreme bankruptcies odds, $r/h=0.6$

Figure 5: extreme bankruptcies odds, $r/h=0.75$

Figure 6: extreme bankruptcies odds, $r/h=0.9$
In figure 4 we see that any number of failure above 6 has a near zero likelihood. For low levels of diversification this results mainly from the higher independence between investors, for high levels of diversification it comes from the fact that with $r/h=0.6$ shocks quickly die out, so that the contagion externality is limited. In figure 5 the scope for contagion is increased as we move to $r/h = 0.75$. In the extreme $r/h=0.9$ case, shocks transmit almost fully across investors, so that the probability of mass failure increases drastically. In particular the odds attached to the “all-fail” outcome reaches up to 15% in when $n=N$. In this case promoting a lower level of diversification amongst investors seems sensible.

The dynamics of the rise in the odds of the all-fail outcome are also interesting. In the $r/h=0.6$ case this likelihood when $n=3$ is only 1% of that when $n=N$, against 7% if $r/h=0.75$ and 41% in the $r/h=0.9$ case. The marginal impact of a rise in $n$ is thus increasing in calm market conditions, but decreasing in adverse ones.

3.1.4 Welfare

We weight the probability attached to a given number of failure against the cost attached to it.

If the cost to society increased in a linear fashion with the number of failures, diversification would unambiguously be desirable from society’s perspective. Yet there are many theoretical reasons for which this cost may in fact grow exponentially with the numbers of defaults. In particular, the well-identified channels for contagion may be enhanced by bankruptcies: we expect surviving investors to become much more risk averse, reputation risk to sky-rock, the liquidity constraint to tighten, etc. Fiordelisi et al (2013) for instance show empirically that bank failures have a strong impact on asset prices. Bernanke (1983) also points out that financial bankruptcies have a more than proportional impact on the real economy, through decreased money supply and increased cost of financial intermediation.

Perhaps due to this high variety of channels and non-linearity, it is hard to estimate the exact cost to society of financial bankruptcies, and authors who want to model this cost have used different mathematical artifices. For instance Ibrahimov et al. (2012) define a time to recovery for the
system, which depends of the number of defaults. We simply specify the following cost function for society:

$$C(\eta) = e^{\beta \eta}$$

where $\eta$, the number of failures, is a random variable, and $\beta$ mitigates the severity of the increase in the cost to society of an additional failure. We show results for values of $\beta \in [0, 0.72]$. We pick this interval because it contains the interesting cases while $\beta = 0.72$ already represents a very high exponentiality of mass failures, as it implies an average cost of a single bankruptcy that is 65 times larger when $\eta = N$ than it is if $\eta = 1$.

The expected cost to society writes:

$$E(C) = \sum_{\eta=0}^{N} P(\eta)C(\eta)$$

Where the probabilities $P(\eta = i)$ were obtained in the previous section. The next 3 figures show this expected cost for our three values of $r/h$, across $\beta$ and $n$. The blue line and the black track the level of diversification for which the expected cost to society for a given $(r/h, \beta)$ couple is the lowest, or highest respectively.

A higher $\beta$ unambiguously works against higher levels of diversification, in which mass failure is more likely. Graphically this is shown by the left turns of the blue line plotting the optimal diversification level.

The impression from the previous section remains. In the $r/h=0.6$ case the contagion externality is too modest for diversification not to be desirable. However with large value of $\beta$ the optimal level of diversification goes surprisingly low, reaching $n=5$. In $r/h=0.75$ the same logic applies, leading the optimal level to $n=3$, which means the cost of an increase in the risk of a large number of failures, and in particular the all-fail outcome, is still acceptable in exchange for the huge private benefits of moving from $n=1$ to $n=2$ and $n=2$ to $n=3$. The contagion externality is larger, leading $n=N$ to become the least desirable level for $\beta > 0.7$. In the last $r/h=0.9$ situation, the scope for contagion is
Figure 7: desirability of diversification with $r/h=0.6$

Figure 8: desirability of diversification with $r/h=0.75$
so high that no individual risk reduction justifies it past $\beta = 0.4$. The perfectly diversified situation becomes the worst possible situation rapidly, when $\beta > 0.2$.

$\beta = 0.7$ implies that the ratio of unit of failure in the “all fail” over that in the “one only” case is about 54, while $\beta = 0.4$ implies a ratio of 3.65. As mentioned this measure is based more on intuition than evidence, we leave it to the reader to assess where its true value would stand. In any case its seems fair to say that, in a situation in which financial shocks propagates linearly, there exist a reasonable set of parameters in which any level of diversification in the economy is dominated by a situation in which investors trade only their own assets, and complete diversification is generally not the optimal level for society.

### 3.2 With possible panic

#### 3.2.1 Change in framework

Remember the demand response to a deviation from fundamentals by convergence trader is given by $h = \frac{m \sigma_{mt}}{E(\sigma_{t+r}^2)}$. We now allow the strength of this demand response to change in the face of extreme selling movement on the markets. Mathematically we capture this through an exogenous fall in $h$, a rise in $r/h$, when sales go beyond a certain threshold $k^*$. In this set-up this could happen because
MT investors review their estimation of the variance $E(\sigma^2_{t,t^*})$ in periods of higher volatility, or because their risk aversion $1/\tau_{mt}$ rises, as they begin to doubt their short-term ability to absorb escalating losses. Somewhat arbitrarily we decide to attribute the rise in $r/h$ to risk aversion. The system now writes:

$$\Delta P_t = \left(\frac{r/h}{n^2} A^T A\right)^{1/2} e_1^F \text{ if } \exists \Delta q_i, |\Delta q_i| > k^*$$

$$\Delta P_t = \left(\frac{r/h}{n^2} A^T A\right)^{1/2} e_1^F \text{ if } \forall \Delta q_i, |\Delta q_i| \leq k^*$$

Where $h^*$ is the discount prevailing in a situation of “panic”. Therefore in this framework investors “panic” when constrained sales get passed $k^*$ on a given market. It may thus only take one extreme movement on one market to trigger panic. Both experience and theory may justify this form: increased risk aversion is highly contagious so that investors operating on the troubled market may lead others to grow more risk averse, accrued counterpart uncertainty is higher when one asset falls drastically than when all fall moderately, margin calls may be triggered when losses exceed a certain level, etc.

We note $\alpha$ the probability that one or more market reaches the sales threshold. It is given by the multivariate cumulative normal distribution $\Phi$ since sales depend linearly on the price shocks at $t=0$, which are normally distributed. We thus have $1 - \alpha = \text{prob}(\forall \Delta q_i, |\Delta q_i| \leq k^*) = \Phi(k^*, ..., k^*)$. The expected value and variance of the sales vector at $t=0$ are respectively 0 and $\Sigma_{\Delta Q} = \sigma^2_{F}(\frac{r}{n^2} A^T A)^2$.

Two things should be noted here: first this technique is consistent with our set-up since the sales are the largest at $t=0$. This means the threshold is either immediately or never reached in our framework, but in both cases $r/h^*$ of $r/h$ remains constant afterward. In other words the initial sales shock “sets the tone” for the rest of the crisis episode. Second, the expected value of $\Delta P_t$ remains zero regardless of the regime, so that the variance with panic is given\(^7\) by $\Sigma_t = \alpha \Sigma^P_t + (1 - \alpha) \Sigma^{NP}_t$.

\(^7\)Using the conditional variance decomposition formula:

$$\Sigma_t = [\Sigma_t/\forall \Delta q_i, |\Delta q_i| < k^*] + \left([\Sigma_t/\forall \Delta q_i, |\Delta q_i| < k^*] + [\Sigma_t/\forall \Delta q_i, |\Delta q_i| > k^*] + [E(\Delta P_t/\forall \Delta q_i, |\Delta q_i| > k^*)]^2 P(\forall \Delta q_i, |\Delta q_i| > k^*) \right] - [E(\Delta P_t)^2]$$

where $E(\Delta P_t) = 0$ and $E(\Delta P_t/\forall \Delta q_i, |\Delta q_i| < k^*) = E(\Delta P_t/\forall \Delta q_i, |\Delta q_i| > k^*)^2 = 0$, as the distribution of $\Delta Q$ is symmetric.
where subscripts P and NP refer to the “panic” and “no panic” cases respectively. Similarly the variances for investors over time is simply the weighted average of covariances matrices in the panic and no panic cases:

\[ \Sigma_l = \alpha \Sigma_l^P + (1 - \alpha) \Sigma_l^{NP} \]  

(9)

3.2.2 Distribution of investors bankruptcy

In the last section we have seen \( r/h=0.6 \) implies a very limited scope for contagion, \( r/h=0.9 \) a large one. These values are thus natural candidates to estimate the no panic and panic case respectively. Regarding the panic threshold, similar to last section we try three values which represent respectively low, moderate, and high tendency for short-term investors to panic, for which panic is triggered for initial shocks of respectively 0.05, 0.01 and 0.005, normalizing \( r=1 \). This represents 5%, 1% and 0.5% of normalized total quantity of each asset\(^8\). Figure 10 to 12 summarizes our findings.

In figure 10, the extreme selling required to trigger panic is very unlikely to happen when \( n>2 \), so that the figure resembles the non-panic \( r/h=0.6 \) case. Extreme failure, particularly the all fail outcome, is very unlikely. On the other hand, when \( n=1 \) the possibility of a panic exists, which may yield a considerable number of failures, with non trivial odds of as much as 80% of investors going under.

\(^8\)These values may appear low but one should remember our time unit is a “day”.

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The intermediate and easy panic cases bring new light to our results. In the moderate panic case, for realistic values of $\beta$ the likelihood of the all-fail outcome is maximized at $n=4$ in the intermediate case, $n=5$ in the easy panic one. In other words, intermediate levels of diversification which were an attractive option without panic now seem particularly harmful, because such levels are not efficient enough in smoothing the wealth shocks faced by portfolio holders, but provide linkages through which shocks may spread across assets. “Medium-low” levels also appear quite undesirable, with odds that $\eta = 9$ and $\eta = 8$ significantly above zero when $n=2$ or $n=3$.

Figures 13 on desirability confirm this intuition: in the intermediate panic case and easy panic one, $n=4$ and $n=5$ become the least desirable option for respectively $\beta = 0.42$ and $\beta = 0.3$, which implies a unit cost of failure in the all fail case over that in the one fail only of 4.38 and 1.48. Past these intermediate levels, more diversification is always desirable, except from $n=9$ to $n=10$ if $\beta$ is very high with easy panic.

Thus high levels of diversification are no longer the least desirable option with a high exponentiality of mass failure, while remaining the most desirable one with a low $\beta$. In the easy panic case
a high $n=9/10$ is optimal if $\beta < 0.36$, implying a ratio of 2.55. Past this level more diversification is still desirable, but no diversification becomes the first-best option. In the intermediate level high levels of diversification remain the most desirable option even with a very high $\beta$. In this case diversification thus provide the first-best allocation if it “goes all the way”, but the worst one if it “stops halfway through”. This suggests that there is a critical threshold level of diversification, past which diversification is desirable, but below which no diversification is the first best option.

These conclusions complement those of Caccioli et al. who find that when leverage is above a critical value, diversification has non-monotonic impact on the odds of a cascade of default, which they define as 5% or more of investors failing.

4 Covariances and extensions

In this section we enter the black box of the covariance matrices between assets prices at a given period $\Sigma_t$, across time $\Sigma_{tot}$, and investors across time $\Sigma_I$, in order to relate the paper to the literature on endogenous risk, and confront our form for covariances to evidence. The discussion on the factors that impact covariances also paves the way for the study the impact or a rise in the number of assets $N$ and a change in the pattern of asset holdings.
4.1 Diagonalization

Drawing the general element of $\Sigma_t$, $\Sigma_{tot}$, and $\Sigma_I$ involves diagonalizing matrices $(A^TA)^t$, $\sum_{t=0}^{\pm \infty} [(r/h)A^TA]^t$, and $\frac{1}{n^2} A \sum_{t=0}^{\pm \infty} [(r/h)A^TA]^t$. Our method is to work our way up from a simple circulant matrix $Z$. The section is mainly technical, the reader interested solely in economic intuition may move immediately to section 4.2.

From the cyclic permutation matrix $Z$ to $A$

$Z$ is the “cyclic permutation matrix”, whose element $z_{i,j}=1$ if $i \equiv j-1 \ [N]$, $z_{i,j}=0$ otherwise. Taking $Z$ to the power $n$ shifts the one-diagonal $n-1$ spots to the right. For instance if $N=5$:

$$Z = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad Z^2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The eigenvalues of $Z$ are readily obtained from the observation that $Z^N = 1^N$, where $1^N$ is the identity matrix. Using the Cayley-Hamilton theorem, the characteristic polynomial of $Z$ is of the form $Q_\lambda = \lambda^n - 1$, and its eigenvalues are the $n$th-roots of unity. Thus the matrix similar to $Z$ is $D_z = \text{diag}(\omega^0, \omega, \ldots, \omega^{N-1})$ where $\omega^k = e^{\frac{2\pi k}{N}} = \cos(\frac{2\pi k}{N}) + i \sin(\frac{2\pi k}{N})$ and $\omega^{-k} = \cos(\frac{2\pi k}{N}) - i \sin(\frac{2\pi k}{N})$ according to Euler’s identity. Note also that since $\omega^N = 1$, we have $\omega^{\alpha(N-k)} = \omega^{\alphaN-\alpha k} = \omega^{-\alpha k}$.

From the properties of $Z$, it appears that any circulant matrix $C$ may be expressed as a polynomial in $Z : C = R(Z)$. This implies $C = R(PD_zP^{-1}) = PR(D_z)P^{-1}$, where $P$ is the change of basis matrix. Therefore:

- $\psi_k$, the $k$th eigenvalue of any circulant matrix $C$ is a polynomial in $\omega^k$, the $k$th eigenvalue of $Z$.

- $P$ is the change of basis matrix for all circulant matrices.

In particular for $A$: 

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\[ A = R(Z) = P\left( \sum_{s=0}^{s=n-1} D_s^Z \right) P^{-1} \quad (10) \]

where \( n \) is the level of diversification. The matrix similar to \( A \) is thus given by \( D_A = \sum_{s=0}^{s=n-1} D_s^Z \).

Its general element is the inverse Fourier transform \( \psi_k = R(\omega_k) = \sum_{s=0}^{s=n-1} \omega^s \), \( \psi_k \) is the sum a geometric series of \( n \) terms and common ratio \( e^{2i\pi k/N} \), so that :

\[ \psi_k = \frac{1 - \omega^{kn}}{1 - \omega^k} = \frac{1 - \cos(\frac{2\pi kn}{N}) - i\sin(\frac{2\pi kn}{N})}{1 - \cos(\frac{2\pi k}{N}) - i\sin(\frac{2\pi k}{N})} \]

except when \( k=0 \), in which case \( \psi_0 = n \). Note that this expression also implies \( \psi_{-k} = \psi_{N-k} \).

**From \( A \) to \( (A^\top A)^t \)**

To move from matrix \( A \) to \( A^\top A \), we use some of the numerous useful properties of the change of basis matrix:

1. \( P \) is unitary: ie \( P^{-1} = P^\dagger \), where \( P^\dagger \) is \( P \)'s conjugate transpose of general term \( \frac{\omega^{-jk}}{\sqrt{N}} \).
2. \( P = P^\top \) and \( P^{-1} = (P^{-1})^\top \), so that \( A^\top = P^{-1}D_A P \)
3. \( P^2 = P^{-2} = V \) where \( V \) is a near reversal matrix.

We may then write \( A^\top A = P^{-1}D_A P P D_A P^{-1} = P^{-1}D_A V D_A P^{-1} \)

Pre-multiplying by \( PP^{-1} \) we get the desired result:

\[ A^\top A = P(VD_A)^2P^{-1} \quad (11) \]

which implies \( (A^\top A)^t = P(VD_A)^{2t}P^{-1} \). The matrix similar to \( (A^\top A)^t \) is thus \( (VD_A)^{2t} \). The covariance between assets price changes at a given period is then \( \Sigma_t = E([\Delta P_t - E(\Delta P_t)][\Delta P_t - E(\Delta P_t)^\top]) = \sigma_F^2\left(\frac{r/h}{n^2}\right)2tP(VD_A)^{4t}P^{-1} \), where we used the fact that \( E(\Delta e_t^F \Delta e_t^F) = diag(\sigma_F^2) \) and that \( (A^\top A)^t \) is symmetric, so that \( P(VD_A)^{2t}P^{-1})^\top = P(VD_A)^{2t}P^{-1} \).

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9 We derive \( P \) in appendix C

10 An example of matrix \( V \) when \( N=4 \) is provided in appendix D
The expression of matrix $V$ implies that $(VD_A)^2$'s element on row $k+1 > 1$ may be expressed as $\phi_k = \phi_{N-k} = \psi_k \psi_{-k} = \frac{1 - \omega^{kn}}{1 - \omega^k} \times \frac{1 - \omega^{-kn}}{1 - \omega^{-k}}$, which will be positive for any $k$. Using (10) and rearranging:

$$\phi_k = \phi_{N-k} = \frac{1 - \cos\left(\frac{2\pi kn}{N}\right)}{1 - \cos\left(\frac{2\pi k}{N}\right)}$$

and $\phi_0 = n^2$ if $k=0$. Note that this implies $\phi_0 = n^2$ is the maximum eigenvalue of $A^\top A$, we thus verify the convergence condition $r < h$ given in section 3.

**Summary of the diagonalization of $A^\top A$:**

- Matrix $Z$'s eigenvalues are the roots of unity.
- Matrix $A$ is a polynomial of $Z$, so its eigenvalues are a polynomial of $Z$'s eigenvalues.
- All circulant matrices have the change of basis matrix $P$ where $PP = V$, $V$ a near reversal matrix.
- This implies $A^\top A = P^{-1}D_AVD_A P^{-1} \implies A^\top A = P(VD_A)^2 P^{-1}$

From $A^\top A$ to $\sum_{t=0}^{\infty} [(\frac{r}{h}) t A^\top A]^t$

$$\sum_{t=0}^{\infty} [(\frac{r}{n^2}) t A^\top A]^t = P(\sum_{t=0}^{\infty} (\frac{r}{n^2}) t (VD_A)^2)^t P^{-1}$$

(12)

The general element of $D_{tot}$ is thus $\xi_k = \sum_{t=0}^{\infty} (\frac{r}{n^2}) t \phi_k)^t$, ie the sum of a geometric series of common ratio $(\frac{r}{n^2}) \phi_k)$. As we consider total price shifts from $t=0$ to $t = \infty$, the formula for the sum of of the terms of geometric series yields:

$$\xi_k = \frac{1}{1 - (\frac{r}{n^2}) \phi_k) t} = \frac{1}{1 - (\frac{r}{n^2} \frac{1 - \cos\left(\frac{2\pi kn}{N}\right)}{1 - \cos\left(\frac{2\pi k}{N}\right)})}$$

\[11\text{see appendix D for a derivation through an } N=4 \text{ example}\]
with \( \xi_0 = \frac{1}{1-r/h} \) since \( \phi_0 = n^2 \). \( \xi_k = \xi_{N-k} \) since \( \phi_k = \phi_{N-k} \).

From \( \Sigma^t_{t=0} \left[ \left( \frac{r}{n^2} \right) A^T A \right]^t \) to \( \frac{1}{n} \Sigma^t_{t=0} \left[ \left( \frac{r}{n^2} \right) A^T A \right]^t \)

Having diagonalized \( A \) and \( AA \), we use (10) and (12) to write:

\[
\frac{1}{n} A \sum_{t=0}^{t=+\infty} \left[ \left( \frac{r}{n^2} \right) A^T A \right]^t = \frac{1}{n} P(D_A)P^{-1}P(D_{tot})P^{-1} = \frac{1}{n} P(D_A D_{tot})P^{-1}
\]

(13)

\( D_A D_{tot} \) is thus the matrix similar to \( A \sum_{t=0}^{t=+\infty} \left[ \left( \frac{r}{n^2} \right) A^T A \right]^t \). The covariance matrix of the vector of total wealth change can thus be re-expressed as \( \Sigma = \frac{\sigma_F^2}{n^2} [PD_A D_{tot}P^{-1}][PD_A D_{tot}P^{-1}]^\top = PD_A D_{tot}V D_A D_{tot}P \).

Post multiplying by \( PP^{-1} \) we obtain \( \Sigma = \frac{\sigma_F^2}{n^2} P(D_A D_{tot}V)^2 P^{-1} \).

The general element of \( (D_A D_{tot}V)^2 \) is \( \rho_k = \psi_k \xi_k \psi_{-k} \xi_{-k} = \phi_k \xi_k^2 \), that is:

\[
\rho_k = \frac{1 - \cos \left( \frac{2\pi k n}{N} \right)}{[1 - \cos \left( \frac{2\pi k}{N} \right)][1 - \cos \left( \frac{2\pi n}{N} \right)]^2}.
\]

4.2 Endogenous covariances

Analytical expression

The diagonalizations of 4.1 yield the following expressions for the covariance between \( k \) and \( j \), which represent respectively asset price change per period, in total, and total investor wealth change:

\[
cov(\triangle p_j, \triangle p_k)_t = \frac{\sigma_F^2}{N} \left( \frac{r}{n^2} \right)^{2t} \left( \sum_{q=1}^{q=(N-1)/2} \cos \left( \frac{2\pi (k-j)q}{N} \right) \left( \frac{1 - \cos \left( \frac{2\pi nq}{N} \right)}{1 - \cos \left( \frac{2\pi q}{N} \right)} \right)^{2t} n^{4t} \right)
\]

(14)
\[
cov(\triangle p_j, \triangle p_k)_{tot} = \frac{\sigma_F^2}{N} \left( \frac{1}{1 - r/h} \right)^2 + 2 \sum_{q=1}^{(N-1)/2} \cos\left( \frac{2\pi (k-j)q}{N} \right) \left( \frac{1}{1 - \left( \frac{r/h}{1 - \cos\left( \frac{2\pi q}{N} \right)} \right)^2} \right) \tag{15}
\]

\[
cov(I_k, I_j)_{tot} = \frac{\sigma_F^2}{N n^2} \left[ \frac{1}{1 - r/h} \right]^2 + 2 \sum_{q=1}^{(N-1)/2} \cos\left( \frac{2\pi (k-j)q}{N} \right) \left( \frac{1 - \cos\left( \frac{2\pi q}{N} \right)}{\left[ 1 - \cos\left( \frac{2\pi q}{N} \right) \right] \left[ 1 - \left( \frac{r/h}{1 - \cos\left( \frac{2\pi q}{N} \right)} \right)^2 \right]} \right) \tag{16}
\]

In the case in which \(N\) is odd\(^{12}\).

These covariances are defined simultaneously and depend upon the same factors: the conditions on the market \(r/h\), the fundamental risk \(\sigma_F\), the distance \(k-j\) in the financial network, the number of assets \(N\), and of course the level of diversification \(n\). As we will see in next section, the two expressions on which we focus, (15) and (16), will in fact move closely together.

What if we allow for panic? Since the covariance matrix is simply the weighted average between that in the linear \(h/r=0.6\) case and the \(h/r=0.9\) one, we have \(\text{cov}(\triangle p_j, \triangle p_k) = \alpha \text{cov}(\triangle p_j, \triangle p_k)^P + (1 - \alpha) \text{cov}(\triangle p_j, \triangle p_k)^N\) for all three forms. The impact of each factor can be split into how it impacts covariances in a given regime and how it impacts the likelihood of each regime.

These expressions are in line with the literature on endogenous risk: the actions of the market participants lead to correlations between assets that would not exist otherwise. Parameters \(\sigma_F^2\) and \(r/h\) in particular can be related to Danielsson et al. and Cont et al., who provide expressions of the covariance matrix that depend on the fundamental covariance structure and some economic variable that govern to the transmissibility. Our network structure allows us to add the “network” parameters \(n, N\) and \((k-j)\).

Note that \(h = \frac{m_{\text{var}}}{E(\sigma^2_{t+1,t^*})}\), where \(E(\sigma^2_{t+1,t^*})\) is the assessment of the individual price variance. If this assessment were correct, ie \(E(\sigma^2_{t+1,t^*}) = \sigma^2_{t+1,t^*}\), then deriving the covariance \(\sigma^2_{t+1,t^*}\) would imply solving a fixed-point problem, as the expression for \(\sigma^2_{t+1,t^*}\) would depend upon itself. However this

\(^{12}\)The expressions change slightly when \(N\) is even, for instance in the case assets prices change per period: \(\text{cov}(\triangle p_j, \triangle p_k) = \frac{\sigma_F^2}{N} \left( \frac{r/h}{n^2} \right)^2 \left( 2 \sum_{q=1}^{(N/2)-1} \cos\left( \frac{2\pi (k-j)q}{N} \right) \left( \frac{1 - \cos\left( \frac{2\pi q}{N} \right)}{1 - \cos\left( \frac{2\pi q}{N} \right)} \right)^2 \right) + n^d + \cos(\pi q) \left( \frac{1 - \cos\left( \frac{2\pi q}{N} \right)}{1 - \cos\left( \frac{2\pi q}{N} \right)} \right)^2 \).
Figure 14: covariances as a function of $n$ for all values of $|j-k|$

fixed point problem is complex, and the question of whether the equilibrium is rational or not is not the main goal of paper. We thus let $E(\sigma_{t+t}^2)$ as a parameter whose value may be $\sigma_{t+t}^2$, or not, noting only that it enter positively in in the expression for $\text{cov}(\Delta p_j, \Delta p_k)$, that is when MT investor expect a high volatility asset prices covariance is higher, as expected.

Comment

We study the effect of $n$, $N$ and $(k-j)$, impact studies of $r/h$ and $\sigma_F^2$ are available on request\textsuperscript{13}. For each factor we first discuss its impact in the $h/r=0.75$ linear model, then that in the easy panic case which in our opinion provides a better description of the reality of crises. We keep the parameter set from section 3.

As could be expected from section 3, \textit{diversification amounts to transferring some variability from the neighboring assets/investors to the more distant ones}. As shown by figure 14, as $n$ rises all assets/investors become more and more related and all covariances converge to a similar value. Note that compared to investors, each asset keeps a larger auto-correlation. This comes from the fact that at $t=0$ its price changes are unrelated, while as $n$ rises portfolios move in more and more similar fashion even at $t=0$.

\textsuperscript{13}Parameter $\sigma_F^2$ rises all covariances uniformly, while a high $h/r$ increases more the covariances and correlations with more remote assets/investors, as it implies that a given shock reaches points that are further away in the network.
Interestingly, in Figure 14 we see that moving from \( n=2 \) to \( n=3 \) involves a higher covariance between \( j \) and \( j+2 \) when \( h/r = 0.75 \) but a lower one when \( h/r = 0.9 \). This reflects the fact that diversification enters negatively through the \( (\frac{r}{h_n}) \) and/or \( (\frac{1}{n^2}) \) terms, and positively through the general element of \( D_{ATA} \) and \( D_{tot} \). In words, on one hand direct connections enhance covariance between assets/investors, on the other in a diversified economy shocks are more spread out. When \( h/r = 0.9 \) and \( n=2 \) the losses on troubled assets snowball to the point where the gain from spreading wealth shocks through diversification exceeds the cost of a direct connection.

Figure 15 describes the panic set-up. As \( n \) rises the wealth shocks and fire sales become weaker, so that the panic threshold is harder to reach. Average covariance falls as the system approaches more and more the \( h/r=0.6 \) case in which transmissibility is lower, but the patterns of convergence remain the same. This figure is consistent with our finding that intermediate levels of diversification are particularly harmful, as such levels provide a unique mix of homogeneity of the covariances and significant likelihood of panic.

The impact of \( N \), which we also consider as an indicator of the completeness of the markets, is contingent on the level of diversification. To see why let us picture the financial network as a circle and a shock stemming from a particular asset as a wave running through it. When a wave has ran through all assets it reaches back that which originated the shock, which we label as “second round” impact. In our home biased network the wave moves slowly from neighbor to neighbor, and thus such “second round” effects have little impact when \( n \) is low, even with \( h/r=0.9 \). However
Figure 16: covariances between $j$ and $k=j+2$ as a function of $N$, for all values of $n$ when $n$ rises and the speed at which the shock completes the circle increases, second round effects become important and losses converge to the same value across assets/investors.

In this case *adding more assets is very desirable from a covariance minimizing perspective as it divides the total fall across more asset/investors*. When $N = +\infty$, ie markets are effectively complete, the covariances tend to 0. Figure 16 shows this by plotting the covariance between assets/investors $j$ and $k=j+2$ for $n \in [1, 10]$.

In the panic case the impact of $N$ becomes a priori ambiguous through a dual effect on the probability of panic. On one hand fire sales are less extreme as each investor is safer, since the covariances between assets in his portfolio are lower. On the other more assets means that the likelihood of a single extreme movement increases. From this perspective figure 17 is interesting. First and foremost the trend is still to a decrease of the covariance between $j$ and $k=j+2$. This means the negative relationship between $N$ and covariances in each regime, similar to figure 16, dominates the impact of $N$ on the probability of panic. This relationship remains monotonic, so that *increasing $N$ stills unambiguously lowers the covariances*.

Nevertheless, the impact of $N$ on the probability of panic changes the patterns of falls. Looking at the highest $n$, the function becomes concave on some segments of $N \in [10, 20]$. On such segments the fall is then lower than its no panic-counterpart, implying that the marginal impact of $N$ on the likelihood of panic is positive. It is logical that this occurs for high levels of diversification since for those levels the “anti-panic” impact of $N$ is low, as investors are already quite insured against shocks.
through diversification. When the likelihood of panic approaches 1 however, both effects on panic die out and the function retakes its “no-panic” evolution. Interestingly, the levels of diversification for which $N$ lowers covariances the most are the intermediate ones, which appeared particularly detrimental from a systemic risk perspective.

The role played by distance also depends on $n$, as when more and more investors/assets become connected the initial position in the network becomes irrelevant. Nonetheless distance is highly explicative for diversification levels below or equal to $N/2$, as Figure 18 shows. For instance in the $n=4$ case asset covariance between direct neighbors is twice that with perfectly distant ones. The relative magnitudes are very similar for $h/r=0.6$ or $h/r=0.9$, so that the panic case gives the same pattern except for the fact that the average covariance falls with $\alpha$ similar to figure 15.

This impact of distance is in line with empirical evidence. Many papers, such as Lane et al. (2004) have demonstrated empirically that informational distance is strongly negatively correlated with bilateral equity holdings. Gravity approaches to estimating correlation usually provide good
Figure 19: Home biased versus non-biased investor

fits. On the link between covariances/correlations and distance during crisis, the evidence is not as definite. This may well be due to the fact that the real impact of a financial crisis is often higher in emerging countries, so that fundamental factors dominate endogenous ones there. Surprisingly little work has been undertaken on correlations during the Subprime crisis, but we may note that by Naoui et al. (2010) who suggest that correlations with the US increased uniformly by about 80% in developed countries during the crisis, while the rise is much variable for emerging ones. Anecdotal evidence that the crisis originated in the US also suggests that distance is important in studying contagion.

This relationship may also be related to the so-called home bias puzzle: since more distant assets have lower covariances, investors apparently forgo diversification benefits by failing to include them in their portfolios. In our model investors not only potentially miss out on lower fundamental covariances, but effectively take on endogenous risk which is at least partly avoidable. Figure 19 shows the variance benefits of following a non biased strategy in our network with home bias.

There are substantial private benefits from deviating from home biases, specially in the early stages of diversification. Such benefits are no longer significant around the N/2 level. At the investor level, accounting for endogenous risk thus increases the incentive to acquire costly information or
pay higher fees.

4.3 Systemic risk impact of a change in the network

The previous section has shown a strong impact of the completeness of markets and distance on covariances. This raises the question of the systemic implications of a wider network, i.e., with a larger $N$, and a non-biased network in which distance becomes irrelevant. For conciseness, we only discuss the easy panic case, as a level of diversification desirable in the most extreme scenario must be desirable also in the other ones.

4.3.1 Optimal network

Let us imagine a network without bias in which each investor picks the portfolio that minimizes his endogenous risk. Taking an $n=3$ example, an investor of type $I$ would hold the asset $i$ he is endowed with and pick the two assets that are the least correlated with his, that is, the assets that other investors of type $I$ hold the least. Through arbitrage, the average quantity by investors of type $I$ of all assets other than their own should thus converge. The sum of these average quantities should be twice the holding of $i$ since $n=3$ and investors ideally spread their holdings equally across assets. In general, the average holding by $I$ should be $1$ of $i$, and $\frac{n-1}{N}$ of other assets.

As we saw in previous sections, such a “home bias free” network enhances the marginal resilience of each investor. However, it also implies that the correlation between all investors becomes homogeneous more rapidly. Swapping networks thus a priori involves weighing up the same costs and benefits that those of increasing the level of diversification: higher contagion costs versus individually sounder benefits. Figure 20 presents the difference between densities in the new network and those in the previous one.

The higher individual resilience is reflected in the fact that the likelihood that all investors survive is higher in the optimal network for all levels of diversification. Yet, the faster contagion cost is also visible: for low levels of diversification this leads to a higher probability of total failure $\eta = N$, though for such levels the increased dependence between investors also brings a strong fall
in the odds of intermediate number of failures. The desirability of moving to an optimal network for low levels of diversification should thus depend on how exponential the cost of financial failures is. For instance in the N=10 case the “optimal” network is preferable only when $\beta < 0.32$ for $n=3$, which means a unit cost of failure that is 1.78 times larger when $\eta = N$ than when $\eta = 1$.

Yet this contagion cost vanishes when $n$ rises as a new effect starts kicking in: in an optimal network sales are more spread across assets which lowers the likelihood of panic. Therefore past $n=4$, moving to an optimal network appears unambiguously desirable. Interestingly, the levels of diversification for which the shift appear the most desirable are the intermediate ones, which were the most dangerous previously this easy panic case.

Another conclusion is that changing qualitatively the network, to account for different heuristics, has a significant impact on our results, as we shall verify in the following section. In Caccioli et al. both of these networks are equivalent, though they in fact have different implications.
4.3.2 Wider network

We move from a $N=10$ network to a $N=20$ one, as in Cont and Wagalath $^{14}$. We choose$^{15}$ to compare only similar absolute levels of diversification, ie $n \in [1, 10]$ even for $N=20$, and normalize failures as the proportion of investors going under, ie $\eta = 10$ with $N=10$ is equivalent to $\eta = (19, 20)$ when $N=20$. Importantly, our choice of absolute $n$ and relative $\eta$ yields the situation in which the desirability of $N = 20$ over $N = 10$ is minimized. The results presented are thus a minima.

One should note that increasing $N$ acts upon the covariance structure, but also has a combinatorial side which may not be relevant here. The $n=1$ shows this combinatorial effect, as with no diversification the problem simply amounts to picking $\eta$ out of $N$ investors. A reassuring sight is that this value of $n$ appears quite independent of the others, which implies the impact of $N$ goes primarily through the covariance structure for $n>1$.

Figure 21 plots the difference between densities with $N=10$ and those with $N=20$.

We see that the likelihood of the extreme “all fail” outcome is higher when $N=10$. Again, the difference is most pronounced for the diversification levels which were previously deemed particularly dangerous. The “all survive” outcome is also more likely. This results from the fact that increasing the number of asset induces more independence across investors, thus reducing the odds of perfectly symmetric situations. In response, the intermediate levels of failure become less likely when $N=10$, with various patterns depending on $n$. When $n$ is fairly low, positive but moderate proportions of investors going bankrupt bear most of the adjustment, which probably reflect the fact that each investor is safer when $N=20$ due to lower covariance across assets. As $n$ rises and this independence naturally falls, the difference becomes more homogenous across all intermediate levels of failures.

In any case, the fact that the “all-fail” outcome is significantly more likely in the $N=10$ case implies that $N=20$ ought to be more desirable, even for a relatively low exponentiality of the cost.

$^{14}$Greenwood and Thesmar use a number of asset of 42, but in this paper the computational burden rises exponentially with $N$, and the simulations on $N=20$ case already took 10 days to complete.

$^{15}$Putting in perspective both cases requires choices. Should we compare the point $(n = 10, \eta = 10)$ when $N=10$ to $(n = 10, \eta = 10)$ or to $(n = 20, \eta = 20)$ when $N=20$?
Figure 21: impact of increasing $N$ on systemic risk to society with the number of failures. This finding is in contrast with that of Caccioli et al. who find an ambiguous impact of rising the $N/n$ ratio.

4.3.3 Wider and better

Figure 22 shows the difference between an optimal and a biased network when $N=20$, to be compared with figure 20 which gives the same information when $N=10$.

Comparing this figure to figure 20 shows that the systemic risk reducing impact of increasing $N$ is higher without home bias. In particular we observe that the magnitude of the increase in the likelihood of the all-fail outcome for low levels of diversification becomes relatively smaller when $N=20$, making the set of parameters for which an optimal network may be undesirable even more trivial. Therefore a “wider” and a more “optimal” network are complementary. This is not surprising with regard to our finding in section 4.2 that increasing $N$ is particularly desirable when the financial shocks spread quickly through the system, which is the case in an optimal network.

We now provide the figure on the desirability of the wider and better network, to be compared
Figure 22: bankruptcies odds, comparison home biased/optimal networks

with the easy panic case in figure 13.

The contrast is evident: the no diversification case remains the most undesirable option for \( \beta \in [0, 0.48] \), which includes value of \( \beta \) for which no diversification was the preferred option in the baseline scenario of figure 13. The “worst” levels of diversification when \( \beta > 0.48 \) remain very low at \( n=2/3 \). The figure also suggests that intermediate levels, which were quick to become the least desirable option, are now quick to become the best one. This finding is in accordance with the significant impact that rising \( N \) and changing the pattern of asset holdings have had on the likelihood of failures.

In the “baseline” easy panic case, systemic risk was maximised around \( n=5 \), past which more diversification was desirable, but such levels were not optimal with \( \beta > 0.36 \). Here the maximum is reached much earlier, and the first-best level of diversification is unambiguously beyond it. In other words when markets are complete and investors are not biased, the window for which diversification may be harmful is much reduced, and the first best allocation is a reachable diversified one. Therefore the threshold past which diversification becomes preferred to no diversification must fall with the completeness of markets and the “efficiency” of asset-holdings.
5 Conclusion

This paper uses a new bottom-up approach to the study of systemic and asset covariances, in line with the emerging endogenous risk literature. Solving the model using circulant matrices we are able to provide a thorough analysis of the impact of diversification on systemic risk, for any possible levels and number of defaults. We find that in equilibrium, diversification increases the probability of mass failure but decreases that of all other non-zero failure outcomes. This leads to diversification to be generally desirable during business as usual periods with low transmissibility of shocks, but not when transmissibility rises and the cost of mass failure rises is high.

However in this paper the link between diversification and systemic risk edges on a deeper question: is contagion driven primarily by human instincts or by hard-wired features of the financial markets? Indeed, as soon as we introduce heuristics by making the strength of the demand for assets contingent on the severity of the initial shock, a new desirable feature of diversification appears. Lower fluctuations to investors wealth brings lower selling movements, minimizing the possibility
of panic.

In this context the relationship between diversification and systemic risk is a concave function whose maximum is reached for intermediate levels of diversification, as such levels create connections between investors without going far enough in minimizing individual risk and the scope for panic. The optimal levels of diversification then appear to be either “high” or “none”, depending on the transmissibility of shocks and on how costly mass failure is, implying there exist a threshold level past which diversification becomes desirable, but below which no diversification should be prefered.

Interestingly this threshold seem to fall with the efficiency of financial markets. When markets are more complete and investors are not biased, the systemic risk maximising level moves from intermediate to low, even in our most pessimistic scenario of “easy panics”. The point at which diversification becomes worth it is reached quickly, so that intermediate levels of diversification actually become the most desirable option.

Let us look at the subprime crisis under this light. Credit backed assets were in fact much more closely fundamentally related than expected by the banks which held them. As these correlations were high, banks were in essence holding a single credit backed asset, their actual diversification level was not high. This led the wealth shocks stemming from adverse movements on ABS to be large, which in turn triggered panic through increased counterpart risk and rising risk aversion. An aggravating factor may have been that international investors tend to be biased towards US securities, the country that has the lowest informational distance to any other country. According to our set-up a lower level of diversification would have been preferable, for risk would not have spread, and a higher diversification would have the first-best allocation, as banks would have been able to digest the losses from ABS markets without triggering panic, while being safer during BAU periods.

Two policy implications may be drawn from this discussion. The first one is quite general: if one believes that transmission of crisis is driven at least partly by heuristic factors, then a key to contain contagion is to maintain confidence. In this case the failure of any institution constitutes
a systemic event, as highlighted by the financial stability board (2009), so that a micro approach may be still be the best option from a policy perspective. Second, in this paper enhancing systemic stability involves *diversifying both more and better*. Some of the progresses in this direction are bound to come from investors themselves, who should start accounting more for endogenous risk of their portfolio, but policy makers may also encourage this trend. In particular the promotion of international diversification though the lowering of information costs and/or taxes appears useful, as it permits both the spreading and the minimization of endogenous risk across assets.

References


Figure 24: lower K and higher $\sigma_F$ in both panic and non panic cases


APPENDIX A

Figure 24 shows the impact of lowering the amount of rising the daily fundamental standard deviation $\sigma_F$ by 25% and capital/threshold for default K by 25% also, which implies moving to a standard deviation of 0.0310 and a capital ratio of 6%. We represent the intermediate linear case $r/h=0.75$ and the easy panic one.

As mentioned in the paper both actions have a comparable impact on systemic risk as they rise the marginal risk attached to each investor. With or without panic, all significant levels of failure are more likely, but the strongest relative increase in on mass failure, which rises sharply with $n$. Note that though we do not provide the figures, in both cases this relative rise in odds is inversely related to the transmissibility $r/h$. For $K$, this is because $r/h$ rises the total variance of portfolio, thus making the increase in the rise in the threshold for default relatively small. For $\sigma_F$ portfolio
are 25% more risk regardless of variance, the evolution in odds thus reflects the difference in the marginal impact of rising the standard deviation on densities.

**APPENDIX B**

The change of basis matrix $P$ is the same for all circulant matrices. We thus look for the change of basis matrix for $Z$.

$$ZP = PDZ \iff \begin{pmatrix} 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_{1,1} & \ldots & p_{1,N} \\ p_{3,1} & p_{3,2} & \ldots & p_{3,N} \\ \vdots & \vdots & \ddots & \vdots \\ p_{N,1} & \ldots & \ldots & p_{N,N} \end{pmatrix} = \begin{pmatrix} p_{1,1} & \ldots & p_{1,N} \\ p_{3,1} & p_{3,2} & \ldots & p_{3,N} \\ \vdots & \vdots & \ddots & \vdots \\ p_{N,1} & \ldots & \ldots & p_{N,N} \end{pmatrix} = \begin{pmatrix} \omega^0 & 0 & \ldots & 0 \\ 0 & \omega & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \omega^{N-2} \end{pmatrix}$$

$$\iff \begin{pmatrix} p_{2,1} & p_{2,2} & \ldots & p_{2,N} \\ p_{3,1} & p_{3,2} & \ldots & p_{3,N} \\ \vdots & \vdots & \ddots & \vdots \\ p_{N,1} & \ldots & \ldots & p_{N,N} \end{pmatrix} = \begin{pmatrix} p_{1,1} p_{1,0} & p_{1,2} p_{1,0} & \ldots & p_{1,N} p_{1,0} \\ p_{2,1} p_{2,0} & p_{2,2} p_{2,0} & \ldots & p_{2,N} p_{2,0} \\ \vdots & \vdots & \ddots & \vdots \\ p_{N,1} p_{N,0} & \ldots & \ldots & p_{N,N} p_{N,0} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & \omega & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \omega^{N-2} & 0 \end{pmatrix}$$

this yields $p_{2,1} = p_{1,1} \omega^0$, $p_{3,1} = p_{2,1} \omega^0 \Rightarrow p_{3,1} = p_{1,1} (\omega^0)^2$, $p_{4,1} = p_{3,1} \omega^0 \Rightarrow p_{4,1} = p_{1,1} (\omega^0)^3$

$p_{2,2} = p_{1,2} \omega$, $p_{3,2} = p_{2,2} \omega \Rightarrow p_{3,2} = p_{1,1} (\omega)^2$, $p_{4,2} = p_{3,2} \omega \Rightarrow p_{4,2} = p_{1,1} (\omega)^3$

$p_{2,N} = p_{1,N} \omega^{N-1}$, $p_{3,N} = p_{2,N} \omega^{N-1} \Rightarrow p_{3,N} = p_{1,N} (\omega^{N-1})^2$, $p_{4,N} = p_{3,N} \omega^{N-1} \Rightarrow p_{4,N} = p_{1,N} (\omega^{N-1})^3$

And so on, the general form is thus: $p_{i+1,j} = (\omega^{j-1})^i p_{1,j}$. Normalizing by $\frac{1}{\sqrt{N}}$ we obtain an orthonormal base for the eigenvectors of the discrete inverse Fourier transform matrix of coefficient $\frac{1}{\sqrt{N}}$, whose general term is $p_{i+1,j+1} = \frac{(\omega^{j-1})^i}{\sqrt{N}}$, with $(i, j) \in \{0, 1, .., N-1\}$:

$$P = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & \omega & \ldots & \omega^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \ldots & \omega^{(N-1)^2} \end{pmatrix}$$
APPENDIX C

$$V_{DA} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_0 & 0 & 0 & 0 \\ 0 & \psi_1 & 0 & 0 \\ 0 & 0 & \psi_2 & 0 \\ 0 & 0 & 0 & \psi_3 \end{pmatrix} = \begin{pmatrix} \psi_0 & 0 & 0 & 0 \\ 0 & \psi_1 & 0 & 0 \\ 0 & 0 & \psi_2 & 0 \\ 0 & 0 & 0 & \psi_3 \end{pmatrix}$$

where $\psi_k = \sum_{s=0}^{n-1} \omega^{ks} = \frac{1-\omega^{kn}}{1-\omega^{k}}$

$$\sigma_{A^2} = (V_{DA})^2 = \begin{pmatrix} \psi_0 & 0 & 0 & 0 \\ 0 & 0 & \psi_3 & 0 \\ 0 & \psi_2 & 0 & 0 \\ 0 & \psi_1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_0 & 0 & 0 & 0 \\ 0 & \psi_1 & 0 & 0 \\ 0 & 0 & \psi_2 & 0 \\ 0 & 0 & 0 & \psi_3 \end{pmatrix} = \begin{pmatrix} \phi_0 & 0 & 0 & 0 \\ 0 & \phi_1 & 0 & 0 \\ 0 & 0 & \phi_2 & 0 \\ 0 & 0 & 0 & \phi_3 \end{pmatrix}$$

where $\phi_0 = (\psi_0)^2$, $\phi_1 = \phi_3 = \psi_3 \psi_1$, $\phi_2 = (\psi_2)^2$

Thus we verify that for $k \in [0, 3]$, $\phi_k = \psi_k \psi_{-k} = \frac{1-\omega^{kn}}{1-\omega^{k}} \times \frac{1-\omega^{-kn}}{1-\omega^{-k}}$, which yields $\phi_k = \frac{1-\cos(\frac{2\pi kn}{N})}{1-\cos(\frac{2\pi k}{N})}$

APPENDIX D

Here we focus on the general element of asset per period covariance. The method is the same for total asset and investors one, only $(\phi_k)^j$ is replaced respectively by $\xi_k$ and $\xi_k \psi_k$. From the expression of $(V_{DA})^2$ in appendix C, and that of $P$ in Appendix B, we may write $\Sigma_i \sigma_F^2 \left( \frac{r/h}{\pi^2} \right)^{2i} P (V_{DA})^{4i} P^{-1}$ as:

$$\Sigma_i = \sigma_F^2 \left( \frac{r/h}{\pi^2} \right)^{2i} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{pmatrix} \begin{pmatrix} \phi_0 & \phi_1 & \phi_2 & \phi_3 \\ \phi_0 & \omega \phi_1 & \omega^2 \phi_2 & \omega^3 \phi_3 \\ \phi_0 & \omega^2 \phi_1 & \omega^4 \phi_2 & \omega^6 \phi_3 \\ \phi_0 & \omega^3 \phi_1 & \omega^6 \phi_2 & \omega^9 \phi_3 \end{pmatrix} = \sigma_F^2 \left( \frac{r/h}{\pi^2} \right)^{2i} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & 1 & \omega^2 \\ 1 & \omega & \omega^2 & \omega^3 \end{pmatrix}$$

Where we used the fact that $\omega^k = \omega^{N-k}$. Using Euler’s identity $\omega^{(k-j)(N-q)} \phi_{N-k} + \omega^{(k-j)q} \phi_k = 2 \cos \left( \frac{2\pi(k-j)q}{N} \right) \phi_k$:

$$(0) = (\phi_0)^{2i} + 2(\phi_1)^{2i} + (\phi_2)^{2i} = (n^2)^{2i} + 2 \left( \frac{1-\cos(\frac{2\pi n}{N})}{1-\cos(\frac{2\pi}{N})} \right)^{2i} + \left( \frac{1-\cos(\frac{2\pi 2n}{N})}{1-\cos(\frac{2\pi 2}{N})} \right)^{2i}$$
\[(1) = (\phi_0)^{2t} + (\omega^3 + \omega^1)(\phi_1)^{2t} + \omega^2(\phi_2)^{2t}\]
\[= (n^2)^{2t} + [\cos(2\pi t) + i\sin(2\pi t) + \cos(2\pi t) - i\sin(2\pi t)](1 - \cos(\frac{2\pi n}{4}))^{2t} + [\cos(\pi) + i\sin(\pi)](1 - \cos(\frac{\pi}{2}))^{2t}\]
\[= (n^2)^{2t} + 2\cos(\frac{2\pi}{4})(1 - \cos(\frac{2\pi}{4}))^{2t} - \left(\frac{1 - \cos(\pi n)}{2}\right)^{2t}\]

\[(2) = (\phi_0)^{2t} + (\omega^2 + \omega^2)(\phi_1)^{2t} + (\phi_2)^{2t} = (\phi_0)^{2t} + (\omega^2 + \omega^{-2})(\phi_1)^{2t} + (\phi_2)^{2t}\]
\[= (n^2)^{2t} + 2\cos(\frac{2\pi}{4})(1 - \cos(\frac{2\pi}{4}))^{2t} + [\cos(2\pi)](1 - \cos(\frac{\pi}{2}))^{2t}\]

The general expression for the covariance between \(k\) and \(j\) given in the paper (for \(N\) even) is verified:

\[\text{cov}(\Delta p_j, \Delta p_k) = \sigma^2 \left(\frac{r/h}{n^2}\right)^{2t} \left(n^{4t} + 2\sum_{q=1}^{N/2-1} \cos\left(\frac{2\pi (k-j) q}{N}\right)\left(1 - \cos\left(\frac{2\pi q}{N}\right)\right)^2 + \cos(2(k-j)\pi)\left(1 - \cos(\pi n)\right)^2\right)\]