Interpretation and limits of sustainability tests in public finance

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Introduction

Public debt sustainability is a major concern at least since the outbreak of the sovereign debt crisis in Europe. Public debt dynamics in countries such as Greece is particularly monitored. But even before the Great Recession, many industrial countries showed persistent deficits and an increasing public debt. Evaluating the sustainability of fiscal policy has naturally come under the spotlight. The Pébereau (2005) and Champsaur-Cotis (2010) administrative reports illustrate this concern for France.

How to define an excessive debt ? A possible definition is given by the IMF (2002): debt has to "satisf[y] the present value budget constraint without a major correction in the balance of income and expenditure given the costs of financing [the government] faces in the market". Obviously it is not straightforward to precisely determine what a "major correction" is. Does it refer to a change in the reaction function of government's revenue and spending to the cycle or the debt, or to a sizeable adjustment of public finances, without any change to the usual reaction function ? If a government already has, in the past, taken successful measures to curb high indebtedness and faces again the same situation, should it be considered insolvent according to the IMF definition ?

In this paper we favor the more usual term of sustainability over solvency. It refers to the ability for a government to pay back its debt with the discounted sum of the primary surpluses generated in the future. As Wyplosz (2007) noticed, the notion of sustainability is essentially forward-looking since it is the future balances that matter. Nevertheless public debt sustainability is often assessed with econometric tests on past data. Boissinot et al. (2004) concluded with standard tests that the French debt was (weakly) sustainable.

Potential behavioural breaks, in the past or between the end of the sample of available data and the near future, represent a first hurdle to interpret these tests' results. From a logical point of view, the only question they give an answer to is : Does the management of public finances as observed in the past justify that investors buy or refuse to buy public debt ? Indeed rational investors buy debt securities only under the condition that the discounted repayments by the government cover the initial debt issuance, i.e. when they believe the government intertemporal budget constraint to hold.

Besides the possible behavioral breaks, Bohn (2007) has underscored that usual tests rely on sustainability conditions that are only sufficient. This weakens their interpretation. As long as investors keep on buying public debt securities, the rejection of a sufficient condition of sustainability can therefore be interpreted in two different ways. Either they expect the government to follow in the future a different policy because it can freely adjust its expenditures and receipts. Or they do base their analysis on the past behaviour of public finances management, and another sufficient condition for sustainability should be tested to justify their buying of public debt.

Following a review of the usual tests and of the sufficient conditions for sustainability they lean on, we will focus on the specification of the government intertemporal budget constraint. Bohn (1995) has stressed that writing this constraint with the interest rate on public debt cannot always be justified. With risk averse lenders and an uncertain economic environment, this constraint relies on a stochastic discount factor which depends on the lenders' preferences. It is possible to imagine theoretical models in which the intertemporal budget constraint is satisfied with the stochastic discount factor but not with the riskless discount factor. We will develop such an example in the second part of this study.

To get round the difficulty related to the specification of the private agents' preferences in empirical analyses, Bohn (1998) suggests to estimate fiscal reaction functions describing how primary surplus reacts to indebtedness. After solving the econometric issues arising when primary surplus and debt have very different persistence, through parametric (Sims, Stock and Watson, 1990) or non-parametric tests (Campbell and Dufour, 1997), we estimate fiscal reaction functions for France and for Greece in the last part of this study.

1 Literature review on usual indicators and tests

1.1 Frequently used indicators

Public debt sustainability analysis often consists first of an economic interpretation of the few key variables entering the debt accumulation equation $d_t = \frac{1+i_t}{1+y_t} d_{t-1} - s_t$, d_t being the end of date t stock of debt divided by GDP, i_t the interest rate, y_t the GDP growth rate and s_t the primary balance over GDP.

An indicator particularly appreciated for its simplicity is the primary balance (before interest payments, prefered to the balance with interests because it is the variable under the short term control of the government) which stabilizes the debt over GDP ratio, given by $s_t^* = \left(\frac{1+i_t}{1+y_t} - 1\right) d_t$, or $s_t^* \approx \left(\overline{i} - \overline{y}\right) d_t$, $\overline{i}$ and $\overline{y}$ being exogenous variables. The intuition is that a country able to stabilize its indebtedness without a major effort can be regarded as safe, a debt reduction requiring only a minimal further improvement of the primary balance. On the contrary, a debt stabilizing primary balance which is reachable only with difficulty signals possible trouble for this country to control its public debt.

In the same spirit the primary balance required to bring back the debt to GDP ratio to 60 % (or whatever level considered safe) over a reasonable period, often 10 or 20 years, may be computed. The study's horizon can also be the very long term, such as in the Public Finances Report 2011 by the European Commission (EC, 2011) focusing on the ageing impact on public finances until 2060. The justification for such an indicator is the following : if the effort to reach within a reasonable horizon a satisfying debt ratio is deemed too large to be credible, then the country's sustainability can be questioned.

These same ideas can be applied to government revenues (or rarely spendings) rather than the surplus to answer the following question : by how much must taxes be raised to stabilize the debt to GDP ratio, or to achieve a balanced budget ? (Blanchard, 1990)

For European countries, it is then possible to compare the effort suggested by these indicators to, for instance, the one each country commits to in the framework of the Stability and Growth Pact. Predictable costs to come can also be taken into account, typically the ageing costs (OECD; EC, 2011), or those stemming from a possible bailout of the banking system (Benassy-Quéré, 2011; EC, 2011). This latter computation is harder to interpret because it involves episodes with very low probability but far-reaching consequences.

All these sustainability analysis are based upon the assumption of exogenous interest and growth rates and they abstract from any feedback fiscal policy may have on these variables (a deficit reduction is not neutral on the growth rate of the economy, either short or long term, nor on the interest rate on which it borrows on the market). A solution to the first remark is to repeat the analysis with different scenarios for the path of the exogenous variables. If sustainability is accepted (respectively rejected) for a scenario convincingly pessimistic (optimistic), one can be more confident in their diagnosis than with only a central scenario. This is the approach chosen by the IMF and the World Bank for their joint *Debt Sustainability Assessments*, which are studied by Wyplosz (2007). They consist in attaching probabilities to different scenarios for the years ahead, 5 in general, and infer a range of possible levels for public debt.

1.2 Economic framework

With D_t the end of period t stock of debt, r_t the interest rate and S_t the primary balance in period t, the identity equation ruling the evolution of the stock of debt is : $D_t = D_{t-1}(1+r_t) - S_t$. Variables can be nominal, real or a ratio to GDP. The interest rate is thus respectively the implied nominal rate i_t paid on debt, the real rate or the rate defined by $1+r_t = (1+i_t)/(1+y_t)$, y_t standing for nominal GDP growth. D_t can be written as a function of expected surpluses with a simplifying assumption on future interest rates. The literature often considers $r_t = r > 0$ constant, or $\mathbb{E}_t[r_{t+1}] = r > 0$ for example. One obtains recursively :

$$D_t = \sum_{i=1}^{N} \frac{\mathbb{E}_t[S_{t+i}]}{(1+r)^i} + \frac{\mathbb{E}_t[D_{t+N}]}{(1+r)^N} \tag{1}$$

With rational lenders, the supply of debt meets a demand if the transversality condition (TC ad hoc) holds. It is the case when debt discounted at a rate r converges to 0,[1] which means lenders expect that debt will be paid

[1]This choice for the discount rate cannot be justified according to Bohn (1995), and it makes the transversality condition and the intertemporal budget constraint ad hoc. This issue will be addressed in section 2.

back in full through discounted expected primary surpluses, which is exactly the intertemporal budget constraint (IBC ad hoc):

$$D_t = \sum_{i=1}^{\infty} \frac{\mathbb{E}_t[S_{t+i}]}{(1+r)^i} \qquad \text{(IBC ad hoc)}$$

$$\lim_{N\to\infty} \frac{\mathbb{E}_t[D_{t+N}]}{(1+r)^N} = 0 \qquad \text{(TC ad hoc)}$$

Both preceding constraints are equivalent, and they are valid when debt is sustainable.

Within this framework it appears that stabilizing the debt to GDP ratio is not a necessary condition for sustainability. Whenever the interest rate is larger than the growth rate, debt can be regarded as sustainable even if the ratio of debt over GDP increases. This is the case when the economy is dynamically efficient.

The relevent choice for the discount rate will be dealt with later. For now we address the interpretation issue of the usual econometric tests : Are they based on sufficient and necessary conditions for sustainability, or only sufficient ones ?

1.3 Usual econometric tests and their limits

The first major contribution in the econometric literature on public finance sustainability is Hamilton and Flavin (1986)[2]. In this article, the interest rate is constant : it is the ex post real interest rate that is earned on one-period government bonds during an average year. The authors test the transversality condition $\lim_{N\to\infty} \frac{\mathbb{E}_t[D_N]}{(1+r)^N} = 0$ against the alternative that the limit exists and is strictly positive : $\mathbb{E}_t \lim_{N\to\infty} D_N/(1+r)^N = A_0 > 0$. Under the alternative and given past data, the agents expect part of the debt never to be paid back.

$$D_t = \mathbb{E}_t \sum_{i=1}^{\infty} \frac{S_{t+i}}{(1+r)^i} + A_0(1+r)^t \qquad (2)$$

If debt is constrained to follow a process of type (2) and if it is possible to show that the process $\{S_t\}$ is stationary, it is then equivalent to test the stationarity of D_t and the nullity of A_0. It is already noticeable that the class of processes considered for $\{D_t\}$ is arbitrarily restricted. Writing $D_t = A_0(1+r)^t + u_t$ where u_t is supposed stationary, D_t and its successive differentiations are not stationary when $A_0 \neq 0$. The null hypothesis ($A_0 = 0$) thus corresponds to D_t stationary, and it is tested against an alternative which rules out any order of integration greater or equal than 1. Rejecting the null of stationarity would therefore not prove that debt is not integrated

[2]These authors make substantial efforts to improve the debt and deficit data (to substract from the deficit the interest payments and the seigniorage revenue for instance, or to deal with the gold stock of the United States), and the notation S_t is therefore slightly different in their paper

of finite order. Hamilton and Flavin find that S_t and D_t are stationary, [3] and infer the nullity of A_0 and the sustainability of American public debt. We will consider later the consequences for the sustainability of public finances of having a debt integrated of order $m > 0$.

Other authors have chosen an approach based on the variables determining debt's variation, such as the deficit or public spending and revenue. Assuming a constant interest rate and that revenue T_t and (without interest) spending G_t are at most $I(1)$, Trehan and Walsh (1988) show that the transversality condition (TC ad hoc) holds if total deficit (interest payments included) is stationary. This condition is equivalent to the existence of a cointegrating relationship between G_t^r (spending with interests) and T_t (revenue).

Quintos (1995) further proves that a $I(2)$ debt is compatible with the transversality condition $\lim_{N\to\infty} \frac{\mathbb{E}_t[D_{t+N}]}{(1+r)^N} = 0$.[4] In this case, debt D_{t+N} behaves, in probability, like a polynomial in N,[5] but this polynomial is asymptotically dominated by the exponentially-growing discount factor. Bohn (2007) broadens this result and proves that the condition for public finances sustainability holds when debt is integrated, whatever its order of integration. From then on the existence of a cointegrating relationship between revenue T_t and spending (including interests) G_t^r ceases to appear as a necessary condition for sustainability. Indeed if these variables are integrated of orders m_T and m_G respectively, but not cointegrated, the order of integration of debt will be m with $m \leq \max(m_1, m_2) + 1$, which ensures the transversality condition $\lim_{N\to\infty} \frac{\mathbb{E}_t[D_N]}{(1+r)^N} = 0$ holds.

Procedures testing if debt is integrated of order m, against an alternative where its order of integration is strictly larger than m, therefore cannot reject the transversality condition. Indeed this condition holds under both the null and the alternative. In a nutshell the null hypothesis corresponds to a condition for sustainability that is only sufficient and not necessary.

The article by Trehan and Walsh (1991) is linked to this literature because it deals with a cointegrating relationship. The null hypothesis chosen by the authors however does not necessarily restrict the order of integration of debt as in the previous tests. Trehan and Walsh (1991) focus on the primary deficit DEF_t and the debt D_{t-1}. They show that the transversality condition holds if these variables are cointegrated (with cointegrating vector

[3]ADF test : rejection of the unit root hypothesis at 10 % (but not 5 %) for debt and primary balance.

The authors test $A_0 = 0$ with a second approach, estimating equation (2) directly leaning on an assumption on the expected primary surpluses. These expectations are first supposed to depend partly on past and present primary balances and take account of lagged debt. They then only depend on past surpluses. In both cases the estimated A_0 is statistically not significant. This article hence deems American public debt sustainable.

[4]This is what Quintos (1995) calls weak sustainability. Boissinot et al. (2004) show that French public finances are weakly sustainable over the period 1978-2002, given that general government's expenditures and receipts are bound by a cointegrating relationship $T_t = \alpha + \beta \cdot G_t^r + \epsilon_t$ with $0 < \beta < 1$.

[5]We use the proof of proposition 1 by Bohn (2007) to write $D_{t+N} = O(N^2)$

$(1, \alpha)$) and if the primary deficit evolves according to $DEF_t = \lambda DEF_{t-1} + \eta_t$ with $\eta_t \sim I(0)$ with zero mean, and $\lambda \in [0, 1+r[$.[6] It follows :

$$D_t - \lambda D_{t-1} = -\left(DEF_{t+1} - \lambda DEF_t\right)/\alpha + \left(\epsilon_{t+1} - \epsilon_t\right)/\alpha \sim I(0)$$

With λ in the interval $]1, 1+r[$, debt is explosive but sustainable because it is discounted by $(1+r) > \lambda$. Bohn (2007) remarks that debt is not integrated in this case, whatever the order of integration. This underscores again that having an integrated public debt is just a sufficient condition for sustainability.

Wilcox (1989) has a special place in the econometric literature on sustainability.[7] As in Trehan and Walsh (1991), debt (in level, nominal or real) is not constrained to be a stationary or an integrated process but it can increase exponentially. The variable of interest is real debt discounted at date t with the realized yield on public debt between a reference year and date t. Wilcox (1989) directly tests the transversality condition $\lim_{N\to\infty} \frac{\mathbb{E}_t[D_N]}{(1+r)^N} = 0$. It is necessary for the transversality condition to hold that discounted real debt is stationary with its unconditional mean equal to zero.[8] Estimating this mean is the purpose of Wilcox's test.

Wilcox (1989) framework does not fully match with the one adopted by Bohn (2007) where it is debt (in level, nominal or real) which is integrated of any order m. It can be shown that debt cannot at the same time be integrated in level and stationary after being discounted (cf. annex A.1). It is however possible to generalize the results proven by Bohn (2007) to a wider class of processes which includes those analyzed by Bohn and Wilcox (proof in annex A.2) :

Proposition 1.1. Let f be a deterministic and discrete function of time.

1. If $D_t/f(t) \sim I(m)$, with $m \geq 0$ and $f(t) \sim o((1+r)^t/t^m)$, then debt verifies the transversality condition (TC ad hoc).
2. If $D_t/f(t) \sim I(0)$ with $f(t) \sim O((1+r)^t)$ and $\mathbb{E}[D_t/f(t)] = 0$, then debt verifies the transversality condition (TC ad hoc).

In particular, whatever $r_0 < r$, $D_t/(1+r_0)^t \sim I(m)$ is a sufficient condition for the ad hoc transversality condition to hold. Furthermore, if $D_t/(1+r)^t \sim I(0)$ with zero expected mean, then the transversality condition also holds : it is the particular case studied by Wilcox (1989). Hamilton

[6] Bohn (2007) notices that these assumptions induce the existence of a fiscal reaction function where the primary balance improves when debt increases : $DEF_t = -\alpha D_{t-1} + \varepsilon_t$ with $0 < \alpha$.

[7] It is not however an isolated contribution. The method described by Wilcox (1989) has recently been adapted by Davig (2005) to allow behavioural breaks in the data generating process for discounted debt.

[8] When discounted real debt is not stationary, two cases are possible : either the conditional mean $\frac{\mathbb{E}_t[D_N]}{(1+r)^N}$ cannot be defined, or it is equal to a random variable not necessarily the null constant.

and Flavin (1986) also belong to this special case, since equation (2) shows that the conditions of the second case are verified when $A_0 = 0$. Finally the first case with a constant function f corresponds to the entire set of processes contemplated by Bohn (2007).

The common characteristic among all the articles described so far is the use of the interest rate on government debt to define the transversality condition and, except Wilcox (1989), to make one of the following assumptions to characterize the evolution of this interest rate :

- $r_t = r > 0$.
- r_t is not autocorrelated and $\mathbb{E}_t [r_{t+1}] = r > 0$.
- r_t is a stationary process with mean $r > 0$.[9]

Beyond the interpretation issues related to the rejection of conditions which are only sufficient that we just highlighted, the previous econometric tests show another weakness, associated with the choice for the discount factor in the intertemporal budget constraint. We examine this fragility in the next section.

2 The Stochastic Discount Factor

2.1 Transversality condition and discount factor in a stochastic environment

Before lending to a government, rational investors make sure that this government is not accumulating debt indefinitely. In other words, they verify that the present value of government debt in a distant future is zero.

In order to define the relevant discount factor, we consider a simplified endowment economy. At each date, a representative agent receives a random endowment Y_t that he cannot store. A fixed proportion g of this endowment is consumed by the government. This public consumption is financed by a tax and the issuance of government bonds to be rapid at a later date. In this decentralized economy, the Euler equation determines the government bond yield ensuring that households effectively consume a fraction $(1 - g)$ of their endowment at each date. In this way, one can price any security contingent on a specific state of nature being realized. We denote by s_t the different states of nature and by $\pi(s_t)$ their probability.

Private agents maximize their intertemporal utility function :

$$\sum_{i=0}^{+\infty} \sum_{s_{t+i}} \beta^i \pi(s_{t+i}) U[C_{t+i}(s_{t+i})] = \mathbb{E}_t \left[\sum_{i=0}^{+\infty} \beta^i U[C_{t+i}] \right]$$

[9]All the results mentioned until now remain valid with such a process. Nevertheless without interests spending G_t should be replaced with an adjusted spending : $G'_t = G_t + (r_t - r)D_{t-1}$. It is the unconditional mean of r_t which enters (TC ad hoc) and (IBC ad hoc).

Given the absence of arbitrage at the optimum, one can determine the price (i.e : how many units of consumption goods) private agents are willing to invest at time t in exchange for one additional unit in the state of nature s_{t+j} at time $(t+j)$:

$$q(s_{t+j}|s_t) = \beta^j \pi(s_{t+j}) \frac{U'[C_{t+j}(s_{t+j})]}{U'[C_t(s_t)]}$$

The yield of a government bond issued at time t and offering a total return $(1+r_t(j))^j$ in every state of nature at time $t+j$ is given by the following formula :

$$\frac{1}{(1+r_t(j))^j} = \sum_{s_{t+j}} \beta^j \pi(s_{t+j}) \frac{U'[C_{t+j}(s_{t+j})]}{U'[C_t(s_t)]} = \mathbb{E}_t \left[\beta^j \frac{U'[C_{t+j}]}{U'[C_t]} \right]$$

In the same way, one can also price government debt at time t with payoff $D_{t+n}(s_{t+n})$ in every state of nature s_{t+n} at time $(t+n)$:

$$\sum_{s_{t+n}} \beta^n \pi(s_{t+n}) \frac{U'[C_{t+n}(s_{t+n})]}{U'[C_t(s_t)]} D_{t+n}(s_{t+n}) = \mathbb{E}_t \left[\beta^n \frac{U'[C_{t+n}]}{U'[C_t]} D_{t+n} \right]$$

The relevant transversality condition in a stochastic environment becomes:

$$\lim_{n \to +\infty} \mathbb{E}_t \left[\beta^n \frac{U'[C_{t+n}]}{U'[C_t]} D_{t+n} \right] = 0 \tag{TC}$$

A strictly positive limit implies that private agents could have a higher intertemporal utility by consuming more at time t and lending less to the government. The country would then refinance itself indefinitely without ever fully repaying the principal as in a Ponzi scheme. Thus, rational investors would not be willing to hold such assets. By contradiction, this shows that the transversality condition (TC) always holds in equilibrium.

This condition is different from the usual transversality condition that is often used in the literature :

$$\begin{aligned}
&\mathbb{E}_t \left[\beta^n \frac{U'[C_{t+n}]}{U'[C_t]} D_{t+n} \right] \\
&= \mathbb{E}_t \left[\beta^n \frac{U'[C_{t+n}]}{U'[C_t]} \right] \mathbb{E}_t \left[D_{t+n} \right] + \mathbb{C}ov_t \left[\beta^n \frac{U'[C_{t+n}]}{U'[C_t]}, D_{t+n} \right] \\
&= \frac{\mathbb{E}_t \left[D_{t+n} \right]}{(1+r_t(n))^n} + \mathbb{C}ov_t \left[\beta^n \frac{U'[C_{t+n}]}{U'[C_t]}, D_{t+n} \right]
\end{aligned}$$

Depending on the sign of the covariance, it could be easier or more difficult for the usual transversality condition to hold. In the case of a deterministic economy or with risk-neutral private agents (with $U[C_t] = \frac{C_t^{1-\epsilon}}{1-\epsilon}$, $\epsilon = 0 \Rightarrow U'(C_{t+n}) = U'(C_t) = 1$), the transversality condition with uncertainty actually boils down to the usual (TC ad hoc). With risk-averse agents in a stochastic economy, both conditions will not in general coincide as it would require D_{t+n} and $\beta^n \frac{U'[C_{t+n}]}{U'[C_t]}$ to be uncorrelated. Of course, it is unlikely to have a zero correlation between these two variables because (marginal utility of) consumption certainly depends on the budgetary and fiscal stand of the government.

The intertemporal budget constraint is now :

$$D_t = \sum_{n\geq 0} \left\{ \frac{\mathbb{E}_t\left[T_{t+n} - G_{t+n}\right]}{(1+r_t(n))^n} + \mathbb{C}ov_t\left(\beta^n \frac{U'[C_{t+n}]}{U'[C_t]}, T_{t+n} - G_{t+n}\right)\right\} \quad \text{(IBC)}$$

2.2 About the example presented by Bohn (1995)

We consider the endowment economy defined previously and examine the case where the government issues debt so that the debt/GDP ratio measured at the end of each period and in every state of nature is constant : $\frac{D_t(s_t)}{Y_t(s_t)} \equiv d$. Taxes are adjusted so that public spending always represents a constant proportion g of the endowment Y_t. Equilibrium in the goods' market therefore implies that agents' consumption is a constant proportion of the endowment at each period: $C_t = (1-g)Y_t$. This fiscal policy will be shown to be sustainable according to the transversality condition in a stochastic setting but not always with the usual one.

We use a CRRA instantaneous utility function with risk aversion denoted ϵ[10] : $U[C_t] = \frac{C_t^{1-\epsilon}}{1-\epsilon}$. The evolution of the endowment is supposed to be log-normal : $\frac{Y_t}{Y_{t-1}} = 1 + y_t$ with $\log(1+y_t) \sim \mathcal{N}(\mu, \sigma^2)$. Finally, we suppose the agents' intertemporal utility to be finite given the properties of this process. Therefore the general term of this positive-term series converges to 0 as t approaches infinity.

$$\lim_{n\to+\infty} \mathbb{E}_t\left[\beta^n U[C_{t+n}]\right] = \frac{(1-g)^{1-\epsilon}}{1-\epsilon} \lim_{n\to+\infty} \mathbb{E}_t\left[\beta^n Y_{t+n}^{1-\epsilon}\right] = 0$$

Given our assumptions, the transversality condition in a stochastic setting holds. Indeed,

$$\lim_{n\to+\infty} \mathbb{E}_t\left[\beta^n \frac{U'[C_{t+n}]}{U'[C_t]} D_{t+n}\right] = \frac{d(1-g)^{-\epsilon}}{U'[C_t]} \lim_{n\to+\infty} \mathbb{E}_t\left[\beta^n Y_{t+n}^{1-\epsilon}\right] = 0$$

We now write the risk-free rate on a loan between date t and date $t+n$:

$$\frac{1}{(1+r_t(n))^n} = \mathbb{E}_t\left[\beta^n \frac{U'[C_{t+n}]}{U'[C_t]}\right] = \mathbb{E}_t\left[\beta^n \prod_{i=1}^{n} (1+y_{t+i})^{-\epsilon}\right]$$

[10]A CRRA utility function is used in this example for the sake of simplicity. It is a well-known fact that this model doesn't allow to reproduce the pattern (i.e. the low values) of the risk-free interest rate with reasonable values of time preference, consumption volatility and risk aversion. But this example only aims at illustrating the differences between the stochastic discount factor and the riskless discount factor in a simple setting. Research in the joint modeling of economic fluctuations and asset returns is still active. We will circumvent the difficulty associated with the relevant specification of the stochastic discount factor in the empirical section of this paper.

We can then compare the riskless rate with the expectation of the debt variable at time $t+n$ in order to see whether or not the usual transversality condition holds:[11]

$$\begin{aligned}\frac{1}{(1+r_t(n))^n} &= \beta^n \exp\left(-\epsilon\mu n + \frac{\epsilon^2\sigma^2 n}{2}\right)\\ \mathbb{E}_t[D_{t+n}] &= dY_t\mathbb{E}_t\left[\prod_{i=1}^{n}(1+y_{t+i})\right] = dY_t\exp\left(\mu n + \frac{\sigma^2 n}{2}\right)\\ \lim_{n\to+\infty}\frac{\mathbb{E}_t\left[D_{t+n}\right]}{(1+r_t(n))^n} &= 0 \Leftrightarrow (\epsilon-1)\mu - (1+\epsilon^2)\frac{\sigma^2}{2} - \log(\beta) > 0\end{aligned}$$

For sufficiently high values of the risk aversion parameter ϵ and of the variance σ^2, the usual transversality condition would be rejected whereas the relevant one in a stochastic setting is always satisfied.

3 Empirical results

Bohn (1998) suggests a way to assess sustainability without being forced to estimate a general equilibrium model and to specify private agents' preferences. He proposes to estimate fiscal reaction functions linking primary surplus and public debt. A positive link is a sufficient condition for sustainability. The theoretical justification for estimating a fiscal reaction function is reminded in appendix A.3. In practice, this method entails econometric difficulties when the persistance of the primary surplus is very different from the persistance of debt. We describe parametric and non-parametric methods in order to deal with these econometric issues and apply these methods on French and Greek data.

3.1 Data description

French national accounts include a financial account for general government from 1978 on. General government debt can be either defined as financial liabilities or as financial liabilities net of financial assets (gross debt or net debt hereafter). Notice that none of these definitions exactly match general government debt as it is defined in the Maastricht Treaty. In this treaty, debt consists in a subcomponent of general government's financial liabilities

[11] With the log-normal hypothesis made for the endowment variation, it can be infered that :

$$\begin{aligned}&\log(1+y_t) \sim \mathcal{N}(\mu,\sigma^2)\\ \Rightarrow &\log\prod_{i=1}^{n}(1+y_{t+i})^{-\epsilon} \sim \mathcal{N}(-\epsilon\mu n, \epsilon^2\sigma^2 n)\\ \Rightarrow &\mathbb{E}_t\left[\prod_{i=1}^{n}(1+y_{t+i})^{-\epsilon}\right] = \exp\left(-\epsilon\mu n + \frac{\epsilon^2\sigma^2 n}{2}\right) = \left(\mathbb{E}\left[(1+y_t)^{-\epsilon}\right]\right)^n\end{aligned}$$

taken at their book value rather than at their market value.[12] It is also worth noticing that debt is never netted from non-financial assets such as land, buildings and infrastructures which are considered more difficult to liquidate if the government needs cash to repay creditors.

Moreover, Reinhart and Rogoff (2010) have constructed long time series of public debt for several countries including France. Reinhart and Rogoff's series for France coincides with general government debt as it is defined in the Maastricht Treaty after 1978. For the period before 1978, their series most likely represents financial liabilities of the central government only. Therefore, we won't use this series before 1978.

From an economic point of view, the most relevant variable seems to be (financial) net debt. Indeed, nationalizations and privatizations from the 1980s and 1990s led to movements of the same sign on the asset and liability sides of the general government balance sheet. Nevertheless, net debt contains without doubt more measurement errors than gross debt and, in particular, Maastricht debt.

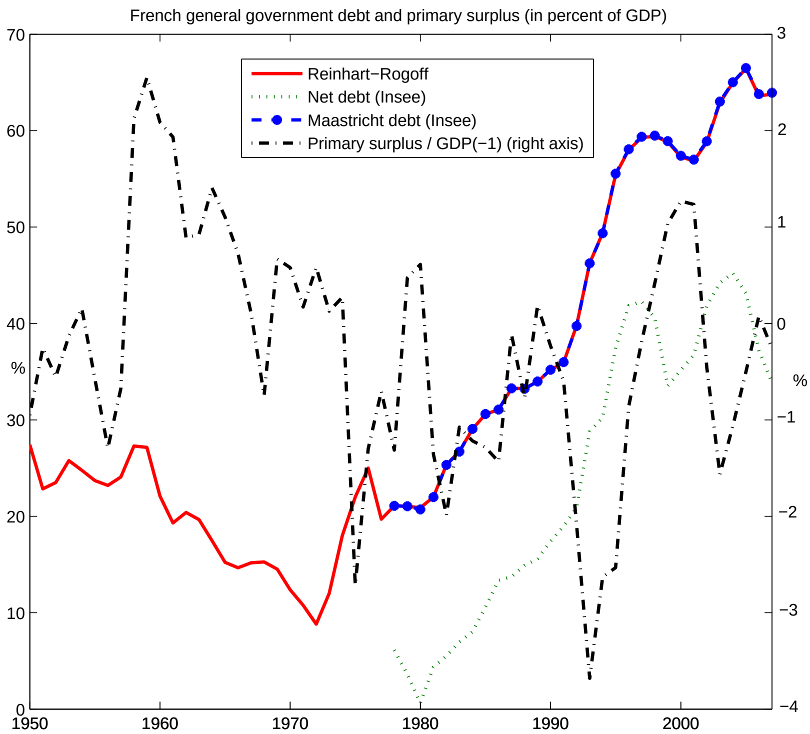

Figure 1: French public debt from 1949 to 2009

[12]General government debt in the sense of the Maastricht Treaty is defined as the sum of total deposits (F2), securities excluding stocks and derivatives (F3 - F34) and credits registered on the liability side (F4) (cf. Bourges 2007). These aggregates are precisely defined in the 1993 System of National Accounts (1993 SNA).

3.2 Estimate of a fiscal reaction function when primary surplus and debt are integrated

When primary surplus and public debt are both integrated time series, the fiscal reaction function is a cointegrating relationship. A finite-sample bias might appear if the evolution of the debt / GDP ratio is correlated to the primary surplus / GDP ratio (cf. 3.3). To eliminate this bias, Stock and Watson (1993) recommend to include leads and lags of public debt's variation in the regression :

$$\frac{S_t}{Y_{t-1}} = \alpha + \beta\frac{D_{t-1}}{Y_{t-1}} + \sum_{i=1}^{n}\gamma_i\left(\frac{D_{t-i}}{Y_{t-i}} - \frac{D_{t-i-1}}{Y_{t-i-1}}\right) + \sum_{j=0}^{n-1}\delta_j\left(\frac{D_{t+j}}{Y_{t+j}} - \frac{D_{t+j-1}}{Y_{t+j-1}}\right) + \varepsilon_t \quad (3)$$

3.3 Estimate of a fiscal reaction function when primary surplus and debt are both stationary, the latter being much more persistent than the former

The following regression has to be estimated :

$$\frac{S_t}{Y_{t-1}} = \alpha + \beta\frac{D_{t-1}}{Y_{t-1}} + \varepsilon_t \quad (4)$$

Results of this regression are difficult to interpret when the primary surplus/GDP ratio is stationary and the debt/GDP ratio also stationary but very persistent[13]. This is a pure econometric issue, not an economic one.

First of all, suppose that the regressor is formally I(1). The error term ε_t is most likely correlated with the evolution of the debt/GDP ratio between the end of period $t-1$ and the end of period t. Indeed, an increase in primary surplus leads to a decrease in debt, everything else held equal. In such a case, the estimator $\widehat{\beta}$ follows a non-standard asymptotic distribution and has a finite-sample bias. The bias is present even when the regressor is predetermined as it is in our case. Thus, it is not a simultaneity bias. Despite the superconvergence property of the estimator, this finite-sample bias is particularly impeding for samples of standard sizes and can lead us to over-reject H_0 ($\beta = 0$) using a Student test with usual critical values.

With a finite sample, the same difficulty arises for time series that are not formally integrated but simply very persistent (cf. Mankiw and Shapiro (1986) for an empirical proof and Banerjee and Dolado (1988) for a theoretical explanation). The sign and the size of this bias depend on the unknown correlation between the error term and the evolution of the debt/GDP ratio.

[13] We assume from the start that unit-root tests do not allow us to differenciate between a formally integrated and a very persistent series for sample sizes. If one is absolutely sure to regress a stationary series on an integrated one, the true value of the β coefficient cannot be different from zero so that the hypothesis on the existence of a fiscal reaction function would have to be rejected.

The existence of this bias casts doubt on the results obtained from panel regressions with numerous countries having a persistent debt/GDP ratio such as those found by Mendoza and Ostry (2010) and the study by the European Commission (2011).

A way to solve that problem is to add an additional lag of the debt/GDP ratio in the regression (4), which gives :

$$\frac{S_t}{Y_{t-1}} = \alpha + \beta \frac{D_{t-1}}{Y_{t-1}} + \gamma \frac{D_{t-2}}{Y_{t-2}} + \varepsilon_t \tag{5}$$

Even when the debt ratio is integrated, estimators $\widehat{\beta}$ and $\widehat{\gamma}$ converge in $\sqrt{T}$ to standard normal distributions centered at β and γ. Indeed, regression (5) can be rewritten differently with these coefficients now associated with stationary variables [14] . Here we use the fact that the difference $\frac{D_{t-1}}{Y_{t-1}} - \frac{D_{t-2}}{Y_{t-2}}$ is stationary. It is a direct application of a theorem by Sims, Stock and Watson (1990). Simulations done by Galbraith and al. (1988) show that using this method yields excellent results in the case of regressors that are not formally integrated processes but simply very persistent.

Then, we can test the existence of a fiscal reaction function of the form (5) and assess the significance of the coefficient β with a standard Student test. A significantly positive β is sufficient to ensure debt sustainability because the error term $\mu_t \equiv \gamma \frac{D_{t-2}}{Y_{t-2}} + \varepsilon_t$ is stationary (cf. Appendix 3).

3.4 Non-parametric tests

The problem arising from the correlation between primary surplus innovations and future values of debt can also be solved using non-parametric tests. According to Campbell and Dufour (1997), if $\frac{S_t}{Y_{t-1}}$ is independent from the past (in particular from $\frac{D_{t-1}}{Y_{t-1}}$ under the null hypothesis $\beta = 0$) and has a median b_0, then the finite-sample exact distribution of the sign statistic $S_g(b_0) = \sum_{t=1}^{n} u[(\frac{S_t}{Y_{t-1}} - b_0)(\frac{D_{t-1}}{Y_{t-1}} - \widehat{m}_{t-1})]$ is known, where $u(z) = 1$ if $z \geq 0$ and $u(z) = 0$ if $z < 0$, and $\widehat{m}_{t-1}$ is the empirical median of the first $t-1$ observations of the debt ratio. Moreover, if the primary surplus ratio has a continuous and symmetric distribution about b_0, then we also know the exact distribution of the *signed rank* statistic $SR_g(b_0) = \sum_{t=1}^{n} u[(\frac{S_t}{Y_{t-1}} - b_0)(\frac{D_{t-1}}{Y_{t-1}} - \widehat{m}_{t-1})]R_t^+(b_0)$ where $R_t^+(b_0)$ denotes the rank of $|\frac{S_t}{Y_{t-1}} - b_0|$ among $|\frac{S_1}{Y_0} - b_0|, \ldots, |\frac{S_n}{Y_{n-1}} - b_0|$ sorted in ascending order, that is $R_t^+(b_0) = \sum_{j=1}^{n} u[|\frac{S_t}{Y_{t-1}} - b_0| - |\frac{S_j}{Y_{j-1}} - b_0|]$.

Both tests rely on the comparison of the signs of $\frac{S_t}{Y_{t-1}} - b_0$ and $\frac{D_{t-1}}{Y_{t-1}} - \widehat{m}_{t-1}$. If β is positive, both primary surplus and debt will tend to be above or

[14] Indeed,

$$\begin{aligned} \alpha + \beta \frac{D_{t-1}}{Y_{t-1}} + \gamma \frac{D_{t-2}}{Y_{t-2}} + \varepsilon_t &= \alpha + \beta \left(\frac{D_{t-1}}{Y_{t-1}} - \frac{D_{t-2}}{Y_{t-2}} \right) + (\beta + \gamma) \frac{D_{t-2}}{Y_{t-2}} + \varepsilon_t \\ &= \alpha + (\beta + \gamma) \frac{D_{t-1}}{Y_{t-1}} - \gamma \left(\frac{D_{t-1}}{Y_{t-1}} - \frac{D_{t-2}}{Y_{t-2}} \right) + \varepsilon_t \end{aligned}$$

below the median at the same time, meaning that $(\frac{S_t}{Y_{t-1}} - b_0)(\frac{D_{t-1}}{Y_{t-1}} - \widehat{m}_{t-1})$ will be more frequently positive than negative. In such a case, the sign statistic $S_g(b_0)$ and the signed rank statistic $SR_g(b_0)$ will be positive and far from 0. In contrast with a negative β, $\frac{S_t}{Y_{t-1}} - b_0$ and $\frac{D_{t-1}}{Y_{t-1}} - \widehat{m}_{t-1}$ will generally display opposite signs, entailing sign and signed rank statistics near from 0.

When the median b_0 of the primary surplus ratio is unknown, Campbell and Dufour (1997) propose two strategies. The first strategy consists in computing the above statistics with the empirical estimator $\tilde{b}_0$ of the median b_0 on the whole sample. However, finite sample distributions of test statistics are not available in this case. The second strategy consists in three steps : first, an exact confidence interval of level α_1 for b_0 is computed; then, test statistics of level α_2 are computed for each value inside the confidence interval; finally, these statistics are combined with the confidence interval for b_0 using Bonferroni's inequality in order to end up with a finite-sample exact non-parametric test at the desired level $\alpha_1 + \alpha_2 = \alpha$.

These non-parametric tests have several advantages compared with the frequently used parametric tests. No restriction is imposed either on the correlation between innovations of the primary surplus ratio and future values of the debt ratio, or on the nature of the innovations generating primary surplus and debt : they can be heteroscedastic and follow non-normal distributions. These tests also rely on exact finite-sample critical values. Numerical simulations done by Campbell and Dufour (1991, 1995) show that these test statistics do not wrongly over-reject the null hypothesis and display a power at least similar to standard t-tests in finite-sample.

However, these non-parametric tests are only valid under the assumption that the primary surplus ratio is not autocorrelated under the null hypothesis. This assumption seems to be more acceptable if we consider the cyclically-adjusted (i.e. structural) primary surplus ratio rather than the non-cyclically adjusted primary surplus ratio. Therefore, we only present results of the non-parametric tests when the dependent variable is the structural primary surplus ratio. Campbell and Dufour (1995) also suggest a method taking into account autocorrelated innovations by considering two subsamples. The first one only contains observations at even dates and the second one those at odd dates. A test of level $\alpha/2$ on each of them will actually amount to an α-level test on the whole sample.

3.5 Empirical results for France

Fiscal reaction functions are only estimated on the 1978-2007 sample so that national accounts data are definitive and output gap estimated are more reliable.

Considering usual stationarity tests (ADF and KPSS), both gross and net debt ratios seem to be I(1) whereas primary surplus ratios, cyclically adjusted or not, seem to be I(0). However, we consider that it is not possible for these tests to distinguish between formally integrated series and stationary but very persistent series with the available data.

Variable	Order of integration	Level		1st difference		2nd difference	
		ADF	KPSS	ADF	KPSS	ADF	KPSS
S /GDP(-1)	0	-3.11**	0.13	-4.26***	0.08	-8.11***	0.08
$Structural\ S$ /GDP(-1)	0	-2.49	0.17	-5.75***	0.07	-9.54***	0.19
D_{net} / GDP	1	-1.58	0.64 † †	-3.31**	0.26	-4.26***	0.36 †
D_{gross} / GDP	1	-1.15	0.68 † †	-3.03**	0.13	-6.47***	0.08

Table 1: Order of integration of fiscal variables (as a % of GDP) with t-stat. *(**) indicates that the null hypothesis of non-stationarity is rejected at a 5% (1%) level and †(††) that the null hypothesis of stationarity is rejected at a 5% (1%) level.

	structural S / GDP(-1)	s.d.	p-value
$D_{net}(-1)$ / $GDP(-1)$	-0.062	0.065	0.345
$D_{net}(-2)$ / $GDP(-2)$	0.055	0.055	0.332
Constant	-0.006	0.004	0.143
Observations	28		
$^{*}p < 0.10$, $^{**}p < 0.05$, $^{***}p < 0.01$			

Table 2: Net debt. Results for France on the 1978-2007 sample : both ratios are considered to be stationary with the debt ratio being very persistent. OLS estimates with Newey-West variances.

	structural S / GDP(-1)	s.d.	p-value
$D_{gross}(-1)$ / $GDP(-1)$	-0.095	0.084	0.266
$D_{gross}(-2)$ / $GDP(-2)$	0.088	0.078	0.267
Constant	-0.004	0.007	0.549
Observations	28		
$^{*}p < 0.10$, $^{**}p < 0.05$, $^{***}p < 0.01$			

Table 3: Gross debt. Results for France on the 1978-2007 sample : both ratios are considered to be stationary with the debt ratio being very persistent. OLS estimates with Newey-West variances.

We estimate the link between cyclically adjusted primary surplus / GDP and debt / GDP ratios considering that both are stationary but the latter very persistent. Indeed, output gap is a potentially important variable determining the primary surplus / GDP ratio and is most likely correlated with the debt to GDP ratio. Bohn (1998) suggests to include it in the regression. Rather than to directly estimate the elasticity of primary surplus to ouput gap because we would need instrumental variables to do it properly, we rely on the elasticity of 0.5 computed for France by Guyon and Sorbe (2009). We use the output gap computed by the European Commission (EC) rather than an HP filter because the estimate of the EC relies on a production function and is therefore more structural. Since most revisions of this series seem to be concentrated on the first 3 or 4 years, the output gap series computed until 2007 is considered to be reliable.

First, we rely on a parametric method using specification (5). Variances of estimators are estimated following Newey and West (1987). Fiscal reaction is not significantly different from 0 whether we use net or gross debt (cf. Tables 2 and 3). However, a left unilateral Student test doesn't allow to accept that the coefficient β is negative [15].

We also apply non-parametric tests introduced by Campbell and Dufour (1995, 1997) to supplement this sustainability analysis. Specification (4) is used and left unilateral tests are computed. The significance of the fiscal reaction coefficient is assessed at a level of 5%. Sign and signed-rank statistics are computed using either the empirical median estimate of the structural

[15] The corresponding t-stat is -0.961, to be compared with tabulated values for a Student distribution with 25 degrees of freedom: -1.058 at 15% and -1.316 at 10%.

primary surplus ratio (median-estimate tests) or a confidence interval for this median (bounds tests).

Results are reported in table 4. Using the empirical median estimate on the sample, the null hypothesis cannot be rejected: p-value is 23% for the sign statistic and 35% for the signed-rank statistic. Using a confidence interval for the median of the structural primary surplus ratio, the null hypothesis is not rejected either (cf. details under table 4). Like Campbell and Ghysels (1995), we then divide the sample in two parts and apply the same non-parametric tests on each subsample so that the assumption of non-autocorrelated residuals becomes more credible. Under the null hypothesis, this assumption means that structural primary surplus ratios are non-autocorrelated in each subsample. These robustness checks confirm our previous results, indicating a lack of response of primary surplus to indebtedness. These results are also in accord with those of the parametric tests.

Finally, if one cannot formally reject the hypothesis of a positive fiscal reaction for France using parametric or non-parametric tests, one cannot either exclude that private investors anticipated a strengthening of the fiscal reaction even before the start of the financial crisis.

3.6 Empirical results for Greece

The evolution of Greek primary surplus and debt to GDP ratios is depicted on figure 2. Greece has been able to reduce its primary deficit rapidly during the 1990s after its debt ratio had started to increase at the beginning of the 1980s. Afterwards, it maintained a positive primary surplus until 2002, allowing to stabilize the debt ratio around 100% of GDP.

Only gross financial debt is available for Greece in international databases. Debt and primary surplus ratios mays both be considered as I(1) for this country. Therefore, the fiscal reaction function is estimated using Stock and Watson (1993) method. Results are presented in table 5. The coefficient on the debt to GDP ratio is estimated to be 0.11 on the 1978-2007 sample and 0.08 on the 1978-2009 sample. It is significant at a 1% level. A Shin (1994) test doesn't reject the cointegration hypothesis between primary surplus and debt ratios. Notice that Greece also appears as a country with very sustainable public finances in Mendoza and Ostry (2008) who also estimate fiscal reaction functions.

Of course, this result may seem confusing when one considers the recent economic developments in Greece. This should be an important warning for the users of econometric sustainability tests. Greece is actually unable to finance its public debt on the market although its past fiscal reaction function points to a sustainable indebtedness. In fact, investors probably anticipated that Greece would be unable to apply this fiscal reaction function at higher debt levels. This is exactly the issue that Bi and Leeper (2012) deal with using a general equilibrium model. Their conclusion is that the default risk does not only depend on a fiscal reaction function but also on the fact that

structural S / GDP(-1)	Median-Estimate Tests				Bound Tests					
	$N-Sg$	Interpretation	$N(N+1)/2-SRg$	Interpretation	$N-SB$		Interpretation	$N(N+1)/2-SRB$		Interpretation
					Q_L	Q_U		Q_L	Q_U	
Whole sample	0.23	H_0 Accepted	0.35	H_0 Accepted	0.93	0.01	Inconclusive	0.64	0.16	H_0 Accepted
Subsample A	0.40	H_0 Accepted	0.37	H_0 Accepted	0.99	0.03	Inconclusive	0.95	0.12	H_0 Accepted
Subsample B	0.61	H_0 Accepted	0.77	H_0 Accepted	0.97	0.09	H_0 Accepted	0.96	0.44	H_0 Accepted

Table 4: Results for France on the 1978-2007 sample. Under H_0, primary surplus ratios and debt ratios are independent. Right unilateral non-parametric significance tests are performed on the statistics $N-Sg$, $N(N+1)/2-SRg$, $N-SB$ and $N(N+1)/2-SRB$, corresponding to left unilateral tests on Sg, SRg, SB and SRB. p-values are indicated in the table. Q_L is the smallest value taken by the test statistic on the confidence interval defined for b_0. Q_U is the largest value.

Note : For median-estimate tests, relying on the empirical median estimate b_0 of the structural primary surplus / GDP(-1) ratio, significance is tested at a 5% level (2.5% for the subsamples).
For bounds tests, a 99% confidence interval $J(0.01)$ is first constructed for the median b_0 on the whole sample (99.5% on each subsample). H_0 is rejected if, for all $b \in J(0.01)$ ($J(0.005)$ for subsamples), the test statistic is above the 4% critical value (2% for subsamples). H_0 is accepted if, for all $b \in J(0.01)$ ($J(0.005)$ for subsamples), the test statistic is less than the 6% critical value (3% for subsamples). It may occur that Q_L is less than the 4% critical value but that Q_U is above the 6% critical value. In this case, test results are said to be inconclusive.

	1978-2007		1978-2009	
	S/PIB(-1)	S structural /PIB(-1)	S/PIB(-1)	S structural /PIB(-1)
$D(-1)$ / $GDP(-1)$	0.110***	0.104***	0.082***	0.075**
	(0.022)	(0.025)	(0.026)	(0.028)
$D(-1)$ / $GDP(-1)$ - $D(-2)$ / $GDP(-2)$	0.055	0.049	0.023	0.012
	(0.117)	(0.133)	(0.138)	(0.154)
D / GDP - $D(-1)$ / $GDP(-1)$	-0.061	-0.052	-0.112	-0.101
	(0.096)	(0.099)	(0.101)	(0.103)
$D(+1)$ / $GDP(+1)$ - D / GDP	0.008	-0.030	-0.150	-0.191
	(0.080)	(0.081)	(0.139)	(0.137)
$D(-2)$ / $GDP(-2)$ -$D(-3)$ / $GDP(-3)$	-0.023	-0.014	-0.032	-0.020
	(0.105)	(0.106)	(0.126)	(0.128)
Constant	-9.236***	-8.791***	-6.679***	-6.170**
	(1.770)	(2.107)	(2.289)	(2.521)
Observations	26	26	28	28
$^{*}p < 0.10$, $^{**}p < 0.05$, $^{***}p < 0.01$				

Table 5: Results for Greece over 1978-2007 and 1978-2009 : surpluses and debt are considered I(1). MCO estimates with Newey-West variances/covariances. Standard deviations in parenthesis

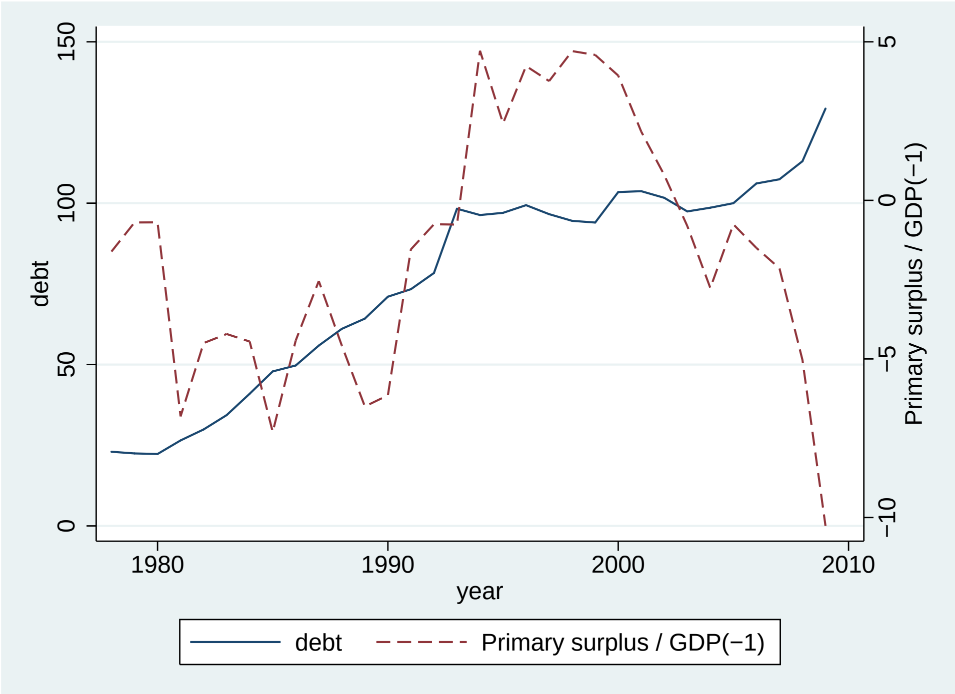

Figure 2: Greek primary surplus and debt ratios over 1978-2009. Data are taken from the AMECO (march 2011) database. Over 1995-2009, they correspond to those published by Eurostat in February 2012.

the product of taxes cannot grow indefinitely to stabilize debt above a certain threshold due to economic and social constraints (Laffer curve).

4 Conclusion

Several indicators may be used in order to assess if public debt is sustainable. Those being commonly used, like the primary surplus stabilizing the debt to GDP ratio, rely on the assumption that interest rates and GDP growth rates are exogenous variables and neglect the feedback effects of fiscal policy. Moreover, commonly used econometric tests have two main drawbacks. First, when they rely on the order of integration of debt or on the estimation of a cointegrating relationship between government receipts and expenditures, they are unable to discriminate between sustainable and unsustainable fiscal policies. Indeed, public debt can be integrated of any order and however sustainable, as shown by Bohn (2007). Second, they often use the ex-post interest rate on government debt as a discount factor. This is theoretically relevant only if there is no uncertainty or if investors are risk-neutral. In general, the government intertemporal budget constraint depends on private agents' preferences and on the interaction between fiscal policy and the rest of the economy.

Bohn (1998) suggests a way to assess sustainability without being forced to estimate a general equilibrium model and to specify private agents' preferences. He proposes to estimate fiscal reaction functions linking primary surplus and public debt. A positive link is a sufficient condition of sustainability. In practice, this method entails econometric difficulties when the persistence of the primary surplus is very different from the persistence of debt. We describe parametric and non-parametric methods in order to deal with these econometric issues.

Fiscal reaction functions have been estimated for France and for Greece using national accounts data over the last 30 years. Because Greece generated an enormous increase of its primary surplus during the 1990s, it appears to fulfill this sufficient condition of sustainability in 2007 and even in 2009. Considering the recent economic developments in this country, this result may seem strange. In fact, investors probably anticipated that Greece would be unable to apply this fiscal reaction function at higher debt levels. This is exactly the issue that Bi and Leeper (2012) deal with using a general equilibrium model. In the case of France, results of parametric and non-parametric tests are more mitigated but the sufficient condition of sustainability cannot be rejected.

Our results highlight the limits of the econometric tests of sustainability. Even if they are correctly specified, they only give an answer to the following question: Is it rational for an investor, using only the past reaction of the primary surplus to debt, to lend money to a government? In fact, fiscal reaction functions may always be different above certain thresholds. Breaks in these functions may also be anticipated by private investors. These tests should always be supplemented by a detailed analysis of the macroeconomic situation in the country and of the way investors form their expectations.

A Annexes

A.1 Proof of the following proposition : debt cannot be at the same time integrated in level and stationary after discounting

There is no non-trivial process which verifies Bohn's and Wilcox's sufficient conditions together.

Indeed if $X_t \sim I(m)$, then $\mathbb{V}[X_t] \sim O(t^{2m})$. This result can be shown by induction, the assertion being trivial for $m = 0$. Let us assume the result established for every $k < m$, and let us consider a process $X_t \sim I(m)$. The process $Y_t = (1 - L)X_t$ can then be defined and it is $I(m-1)$. The variance of $X_t = X_0 + Y_t + ... + Y_0$ equals $\sum_{j=0}^{t} \mathbb{V}[Y_j] + \sum_{j=0}^{t} \mathbb{C}ov[X_0, Y_j] + \sum_{j_1,j_2=0,j_1 \neq j_2}^{t} \mathbb{C}ov[Y_{j_1}, Y_{j_2}]$. The first term is a sum over $1 \leq j \leq t$ of $O(j^{2m-2})$ terms thanks to the induction hypothesis, so it is $O(t^{2m-1})$. The second term, using Cauchy-Schwarz inequality, is a sum of $O(j^{m-1})$ terms, and therefore is $O(t^m)$. The last term, still with Cauchy-Schwarz inequality, is a sum on $1 \leq j_1 \neq j_2 \leq t$ of $O(j_1^{m-1} j_2^{m-1})$ terms, thus is $O(t^{2m})$. Eventually we did prove that the variance of the process X_t is $O(t^{2m})$.

Furthermore, with $Y_t \sim I(m')$ for $m' \leq m$, Cauchy-Schwarz inequality allows to write $\mathbb{C}ov[X_t, Y_t] \sim O(t^{2m})$. Using $\Delta(uv) = \Delta(u)v + u\Delta(v) + \Delta(u)\Delta(v)$, one can easily establish with mathematical induction on $k \geq 0$ that there exists coefficients $\{\alpha_{k,j}\}_{0 \leq j \leq k}$ such that :

$$\Delta^k \left[\frac{D_{t+n}}{(1+r)^{t+n}} \right] = \frac{1}{(1+r)^{t+n}} \sum_{j=0}^{k} \alpha_{k,j} \Delta^j D_{t+n}$$

Developing for $k \geq 0$:

$$\mathbb{V}\left[\Delta^k \left[\frac{D_{t+n}}{(1+r)^{t+n}} \right] \right] = \frac{1}{(1+r)^{2(t+n)}} \mathbb{V}\left[\sum_{j=0}^{k} \alpha_{k,j} \Delta^j D_{t+n} \right]$$

And we notice that the term whose variance we look at is a sum of $k+1$ terms all integrated of order smaller or equal than m. Developing the variance will allow to write it as a sum of $(k+1) + k(k+1)/2$ terms which are all $O((t+n)^{2m})$. We can then infer that $\mathbb{V}\left[\Delta^k \left[\frac{D_{t+n}}{(1+r)^{t+n}} \right] \right] \sim O((t+n)^{2m}/(1+r)^{2(t+n)})$, which means that for every $k \geq 0$, this variance will converge towards 0 when n goes to infinity and therefore cannot be a strictly positive constant, which is one of the conditions for the k times differentiated discounted debt to be stationary and non-trivial.

A.2 Proof of proposition 1.1

The proof is very close to that of proposition 1 in Bohn (2007). Noting $d_t = D_t/f(t)$, this d_t verifies the integration condition of the variable called B_t by Bohn (and which is D_t here). With Bohn's notations :

$$d_{t+n} = \sum_{k=0}^{m-1} p_k(n) \Delta^k d_t + \sum_{i=1}^{n} p_{m-1}(i) \Delta^m d_{t+(n+1-i)}$$

This implies:
$\mathbb{E}_t[D_{t+n}]/(1+r)^n = (1+r)^t f(t+n)/(1+r)^{t+n}(Q(n) + n^m \mathbb{E}_t[Y_t(n)])$.

$Q(n)$ is a polynomial of order $m-1$, so is $O(n^{m-1})$. With the stationarity of $\Delta^m(D_t/f(t))$, we get that $\mathbb{E}_t[Y_t(n)]$ is bounded, so that the term in parenthesis is $O(n^m)$. With the hypothesis made on $f(t)$ at infinity, we conclude that (TC ad hoc) holds.

A.3 Sufficient condition of sustainability based on the fiscal reaction function

This appendix is a reprint of the proof by Bohn (1998) that differs only by a slight change in date conventions for the public debt variable : D_t is now the debt at the end of period t.

The public debt accumulation equation (nominal or real) is : $D_t = (D_{t-1} - S_t)(1 + R_t)$.

This equation can be divided by the GDP of period t :

$$d_t \equiv \frac{D_t}{Y_t} = \left(\frac{D_{t-1}}{Y_{t-1}} - \frac{S_t}{Y_{t-1}}\right) \cdot (1 + R_t) \frac{Y_{t-1}}{Y_t} = \left(\frac{D_{t-1}}{Y_{t-1}} - \frac{S_t}{Y_{t-1}}\right) \cdot x_t$$

Suppose that the fiscal reaction function is of the following form, with $0 < \rho < 1$:

$$s_t \equiv \frac{S_t}{Y_{t-1}} = \rho \frac{D_{t-1}}{Y_{t-1}} + \mu_t$$

.

The fiscal reaction function we consider is slightly different from the one postulated by Bohn (1998) because the primary surplus of period t is divided by the GDP of $t-1$. This minor correction originates in the change of date convention for the public debt.

The primary surplus can then be taken out of the equation governing the evolution of the public debt. By successive iterations, we get :

$$d_{t+n} = \left(\prod_{j=1}^{n} x_{t+j}\right)(1-\rho)^n d_t - \sum_{i=1}^{n}\left(\prod_{j=1}^{n} x_{t+j}\right)(1-\rho)^{n-i} \mu_{t+i}$$

The stochastic discount factor is defined using the private agents' preferences : $u_{t,n} \equiv \beta^n \frac{U'(C_{t+n})}{U'(C_t)}$

The risk-free interest rate can then be deduced from that stochastic discount factor : $\mathbb{E}_t\left[u_{t+i,1} \cdot (1 + R_{t+i+1})\right] = 1$

We deduce in the next steps :

$$\frac{\mathbb{E}_t\left[u_{t,n}\cdot D_{t+n}\right]}{Y_t}=\mathbb{E}_t\left[u_{t,n}\cdot\prod_{j=1}^{n}\left(1+y_{t+j}\right)\cdot d_{t+n}\right]$$

$$=(1-\rho)^n\cdot\mathbb{E}_t\left[u_{t,n}\cdot\prod_{j=1}^{n}\left(1+y_{t+j}\right)\cdot\prod_{j=1}^{n}x_{t+j}\right]\cdot d_t$$

$$-\sum_{i=1}^{n}(1-\rho)^{n-i}\cdot\mathbb{E}_t\left[u_{t,n}\cdot\prod_{j=1}^{n}\left(1+y_{t+j}\right)\cdot\prod_{j=i}^{n}x_{t+j}\cdot\mu_{t+i}\right]$$

$$=(1-\rho)^n\cdot\mathbb{E}_t\left[u_{t,n}\cdot\prod_{j=1}^{n}\left(1+R_{t+j}\right)\right]\cdot d_t$$

$$-\sum_{i=1}^{n}(1-\rho)^{n-i}\cdot\mathbb{E}_t\left[u_{t,n}\cdot\prod_{j=i}^{n}\left(1+R_{t+i}\right)\cdot\prod_{j=1}^{i-1}\left(1+y_{t+j}\right)\cdot\mu_{t+i}\right]$$

$$=(1-\rho)^n\cdot\mathbb{E}_t\left[\prod_{j=1}^{n}u_{t+j-1,t+j}\cdot\left(1+R_{t+j}\right)\right]\cdot d_t$$

$$-\sum_{i=1}^{n}(1-\rho)^{n-i}\cdot\mathbb{E}_t\left[u_{t,i-1}\cdot\prod_{j=1}^{i-1}\left(1+y_{t+j}\right)\cdot u_{t+i-1,n}\cdot\prod_{j=i}^{n}\left(1+R_{t+i}\right)\cdot\mu_{t+i}\right]$$

$$=(1-\rho)^n\cdot d_t-\sum_{i=1}^{n}(1-\rho)^{n-i}\cdot\mathbb{E}_t\left[u_{t,i-1}\cdot\prod_{j=1}^{i-1}\left(1+y_{t+j}\right)\cdot\mu_{t+i}\right]$$

By assumption, the discounted value of future revenues $\sum_{k=1}^{n}Y_t\cdot\mathbb{E}_t\left[u_{t,k}\cdot\prod_{j=1}^{k}\left(1+y_{t+j}\right)\right]$ is finite.

Therefore : $\lim_{k\to+\infty}\mathbb{E}_t\left[u_{t,k}\cdot\prod_{j=1}^{k}\left(1+y_{t+j}\right)\right]=0.$

In addition, the process μ_t is stationary so bounded in probability, implying : [16] $\lim_{k\to+\infty}\mathbb{E}_t\left[u_{t,k}\cdot\prod_{j=1}^{k}\left(1+y_{t+j}\right)\cdot\mu_{t+k}\right]=0$

The existence of such fiscal reaction function is sufficient for the transversality condition to hold, whatever the exact form of the stochastic discount factor :

$$\lim_{n\to+\infty}\frac{\mathbb{E}_t\left[u_{t,n}\cdot D_{t+n}\right]}{Y_t}=0$$

[16] This part of the proof by Bohn (1998) is not so obvious. Moreover, it is worth reminding that Bohn (1998) considered a stationary process μ_t in his paper, except in the appendix where he detailed his proof and viewed μ_t to be a strictly bounded process. It is possible to extend the proof by noticing that the assumption made on the discounted value of future revenus implies $u_{t,k}\cdot\prod_{j=1}^{k}\left(1+y_{t+j}\right)\underset{k\to+\infty}{\overset{\mathbb{P}}{\to}}0$

A.4 Cumulative distribution function of the French structural primary surplus / GDP(-1) ratio

Figure 3 plots the histogram and the cumulative distribution function of the structural primary surplus / GDP(-1) ratio which is more or less symmetric with respect to its median of -1.01 on the considered time window. Using the non-parametric signed rank statistic is therefore possible.

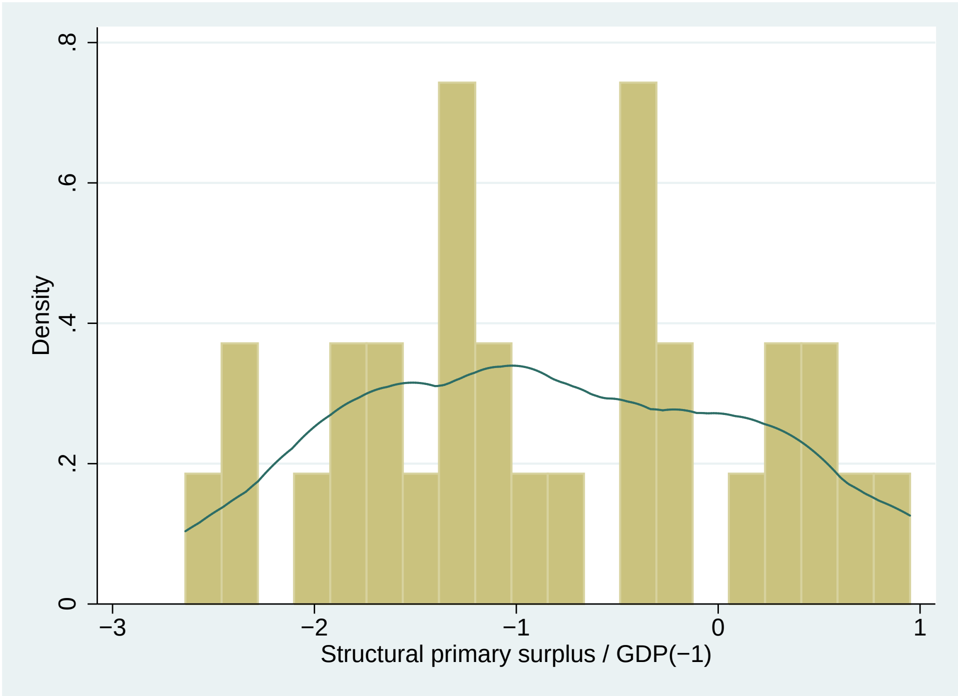

Figure 3: Cumulative distribution function of the French structural primary surplus / GDP(-1) ratio on the period 1978-2007

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