A TIME SERIES MODEL OF INTEREST RATES WITH THE EFFECTIVE LOWER BOUND

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the Federal Open Market Committee,
the Bank for International Settlements,
or their respective staffs.

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NOMINAL INTEREST RATE
U.S. DATA
3m Tbill, quarterly avg., APR
When nominal interest rates are near zero . . .

- How to do time series with ELB constraint?
  - Forecasting / nowcasting
  - Trend-cycle decomposition: What is $\bar{r}_t$?
  - Impulse responses
  - . . .

- Conventional models not equipped to deal with ELB
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Shadow rate model

- Hypothetical nominal rate, unconstrained by lower bound
- Our approach: time series w/o no-arbitrage
Shadow Rate $s_t$

Nominal interest rate that would prevail in the absence of lower bound constraint

Observed Rate $i_t$

$$i_t = \max (s_t, ELB)$$
Shadow Rate $s_t$

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$$i_t = \max (s_t, ELB)$$

Key idea of our project:

Model $s_t$ with typical time-series tools and handle $\max$ operator
**Term Structure Models**

\[ i_t = \max (s_t, ELB) \text{ is a payoff} \]

- Krippner, Wu & Xia, Bauer & Rudebusch, ...  
- No-arbitrage conditions pin down dynamics of \( s_t \)  
- Time-invariant, affine processes  
- Difficult: time-varying dynamics, changes in parameters
**Term Structure Models**

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**Time Series**

\[ i_t = \max (s_t, ELB) \text{ is a censoring function} \]

- Agnostic about asset pricing
- Time-series projections pin down \( s_t \)
- Can do time-varying parameters, stochastic volatility, etc.
- Resurrects many time series models at ELB
• Identical to actual rate when above ELB
• Identical to actual rate when above ELB

• **Identified by historical co-movements**
  between actual rates and other variables
• Identical to actual rate when above ELB
• Identified by historical co-movements between actual rates and other variables
• **At ELB: a latent state variable** that characterizes the dynamics of actual rates and other variables
• Identical to actual rate when above ELB
• Identified by historical co-movements between actual rates and other variables
• At ELB: a latent state variable that characterizes the dynamics of actual rates and other variables
• **At ELB: Projected “lever” of monetary policy**, based on macro variables, longer-term yields *and* constrained level of actual rate
## RELATED LITERATURE

### Macro-Time Series at the ELB


### Dynamic Term-Structure Models


### Unobserved Component Models of the Macroeconomy

AGENDA

1. An Unobserved Components Model
2. Estimates of Shadow Rates, Trends and Cycles
3. Interest Rate Forecasts
4. Impulse Response Analysis
Let’s model the following variables:

- $i_t$: 3m Tbill Rate
- $y_t$: vector of 2-year, 5-year and 10-year Treasury yields
- $\pi_t$: PCE headline inflation
- $\tilde{c}_t$: output gap (CBO)

We then need to capture:

- Great Inflation, Great Moderation, Great Recession
- Time-varying volatility
- Drifting means in inflation, nominal yields . . .
- The effective lower bound on nominal rates
Beveridge-Nelson trend as unobserved component

Data: \[ X_t = \bar{X}_t + \tilde{X}_t \]

Gap: \[ \tilde{X}_t \sim I(0) \]

Trend: \[ \bar{X}_t = E_t X_{t+\infty} \]
Beveridge-Nelson trend as unobserved component

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Trend: \( \bar{X}_t = E_t X_{t+\infty} = \bar{X}_{t-1} + \sum_{t}^{1/2} \bar{\epsilon}_t \)

Stochastic volatility in trend shocks

- Data can be strongly trending or nearly stationary
- Time-varying persistence
<table>
<thead>
<tr>
<th>Potential output</th>
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**Headline inflation**

\[ \pi_t = \bar{\pi}_t + \tilde{\pi}_t \]

**Nominal (shadow) rates**

\[ s_t = \bar{s}_t + \tilde{s}_t \]

**y_t**

\[ y_t = \bar{s}_t + \bar{p}_t + \tilde{y}_t \]

**spreads:**

\[ y_t - s_t \sim I(0) \]

\[ \bar{s}_t = \bar{\pi}_t + \bar{r}_t \]

**Trend dynamics**

\[ \bar{\pi}_t = \bar{\pi}_t - 1 + \sigma_{\pi,t} \epsilon_{\pi,t} \] (Baseline)

\[ \bar{r}_t = \bar{r}_t - 1 + \sigma_{r} \epsilon_{r,t} \]
### Potential output

From CBO

### Headline inflation

\[ \pi_t = \bar{\pi}_t + \tilde{\pi}_t \]
## TRENDS AND COINTEGRATION

### Potential output
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\[ \bar{s}_t = \bar{\pi}_t + \bar{r}_t \]
\[ \bar{\pi}_t \perp \bar{r}_t \]
### Potential output

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### Trend dynamics

\[ \bar{\pi}_t = \bar{\pi}_{t-1} + \bar{\sigma}_{\pi,t} \bar{\varepsilon}_{\pi,t} \]
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**Trend dynamics**

\[ \bar{\pi}_t = \bar{\pi}_{t-1} + \bar{\sigma}_{\pi,t} \tilde{\epsilon}_{\pi,t} \quad \text{(Baseline)} \]
\[ \bar{r}_t = \bar{r}_{t-1} + \bar{\sigma}_r \tilde{\epsilon}_{r,t} \]
### Trend vs. Natural Real Rate

#### Trend real rate \( \bar{r}_t = E_t r_{t+\infty} \)

- Forecast of real (shadow) rate in the very long-run
- Agnostic about “appropriate” level of current real rate
- Long-run restriction: applicable to many models
- Equal to median of actual real rate in long run if ELB binds only occasionally

#### Natural rate, a.k.a. neutral rate, a.k.a. \( r^*_t \)

- Benchmark for current policy
- Typically derived within context of specific structural assumptions to identify “policy-relevant” frictions

### If frictions have no long-run effects:

\[
\bar{r}_t = E_t r^*_{t+\infty}
\]
Generic VAR w/SV

\[ \tilde{X}_t = \begin{bmatrix} \tilde{\pi}_t & \tilde{c}_t & \tilde{s}_t & \tilde{y}^2_t & \tilde{y}^5_t & \tilde{y}^{10}_t \end{bmatrix} \]

\[ A(L) \tilde{X}_t = B \tilde{\Sigma}_t^{1/2} \varepsilon_t \]

where \( B \) unit-lower-triangular and \( \tilde{\Sigma}_t = \text{diag} (\tilde{\sigma}_t^2) \)

SV in VAR residuals

\[ \log (\tilde{\sigma}_t^2) = (I - \rho)\mu + \rho \log (\tilde{\sigma}_{t-1}^2) + \Phi^{1/2} \eta_t \]

\( \rho \) diagonal, \( \eta_t \sim N(0, I) \) and \( \Phi \) dense

(similar AR1 for trend SV)
State transition

\[ \xi_t = [\bar{X}_t' \; \tilde{X}_t' \; \ldots]' = A_t \xi_{t-1} + B_t \varepsilon_t \]

Shadow-rate "measurement" equation

\[ X_t = \begin{bmatrix} S_t \\ M_t \end{bmatrix} = C_t \xi_t \]

Actual-rate measurement equation

\[ Z_t = \begin{bmatrix} Y_t \\ M_t \end{bmatrix} = \begin{bmatrix} \max (S_t, ELB) \\ M_t \end{bmatrix} \]
### State transition
\[
\xi_t = \begin{bmatrix} \tilde{X}'_t & \tilde{X}'_t & \ldots \end{bmatrix}' = A_t \xi_{t-1} + B_t \varepsilon_t
\]

### Shadow-rate “measurement” equation
\[
X_t = \begin{bmatrix} S_t \\ M_t \end{bmatrix} = C_t \xi_t
\]

### Actual-rate measurement equation
\[
Z_t = \begin{bmatrix} Y_t \\ M_t \end{bmatrix} = \begin{bmatrix} \max (S_t, ELB) \\ M_t \end{bmatrix}
\]

**Estimated with Bayesian MCMC sampler**
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Our model implies a reaction function . . .

\[ s_t = \bar{r}_t + \bar{\pi}_t \]
\[ + \phi_c \tilde{C}_t + \phi_\pi (\pi_t - \bar{\pi}_t) + \phi_s (s_{t-1} - \bar{s}_{t-1}) \]
\[ + \ldots + \varepsilon^m_t \]
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LONG-RUN REAL RATE
Grey: median, 50%, 90% bands

[Graph showing a trend over time from 1960 to 2017 with smoothed median, 50%, and 90% bands for the long-run real rate.]
LONG-RUN REAL RATE
Grey: median, 50%, 90% bands. Red: $i_t - E_t \pi_{t+1}$
LONG-RUN REAL RATE
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LONG-RUN REAL RATE
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LONG-RUN REAL RATE
Our estimates indicate a much smaller decline than others . . .

1) Stochastic volatility in trends and gaps

SV allows our model to adjust the signal-to-noise ratio as amplitude of business cycle changes over the course of Great Inflation / Moderation / Recession

2) Shadow rate keeps moving throughout ELB period

Model sees ongoing cycle as opposed to “ELB = nominal trend”
1) Business cycle measure
Similar results w/ CBO unemployment rate gap

2) Ordering of variables in gap VAR

\[ A(L) \tilde{X}_t = B \tilde{\Sigma}_t^{1/2} \varepsilon_t \]

- VAR-SV not invariant to ordering of variables
- Similar results with various orderings for \( \tilde{X}_t \)

3) SV in \( \bar{r}_t \)

\[ \bar{r}_t = \bar{r}_{t-1} + \bar{\sigma}_{r,t} \bar{\eta}_{r,t} \quad \log \left( \bar{\sigma}_{r,t}^2 \right) \sim AR(1) \]

- MDD: harmonic mean and particle filter for \( p(Z|\theta) \)
- Bayes factors strongly prefer constant variance in \( \bar{r}_t \)
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FORECAST COMPARISON

Quasi-real-time forecasts

- Compare to model of Wu and Xia (2016) and SPF
- Mean/median forecasts compared with RMSE and MAD
- Rel. RMSE $> 1$: our model performs better

Relative RMSE (post 2008):

<table>
<thead>
<tr>
<th>Forecast horizon $h$ (quarters)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<td>$W-X$ (short rate)</td>
<td>0.68</td>
<td>0.94</td>
<td>1.01</td>
<td>1.10</td>
<td>1.44</td>
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<tr>
<td>SPF (short rate)</td>
<td>0.36</td>
<td>1.03</td>
<td>1.26</td>
<td>1.30</td>
<td>1.25</td>
</tr>
<tr>
<td>$W-X$ (long rate)</td>
<td>0.99</td>
<td>1.03</td>
<td>1.06</td>
<td>1.10*</td>
<td>1.12**</td>
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<tr>
<td>SPF (long rate)</td>
<td>0.67***</td>
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<td>1.25**</td>
<td>1.35***</td>
<td>1.50***</td>
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Stars indicate Diebold-Mariano test significance levels
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Comparable to SVAR literature:

- VAR(∞) representation for $X_t = [S'_t \ M'_t]'$
- Policy shock is linear combination of VAR residuals
- CEE-like Choleski scheme: MP shock is . . .
  - shadow-rate surprise
  - orthogonal to $\pi_t$ and $\tilde{c}_t$
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UCSV generates time-varying VAR
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  - orthogonal to $\pi_t$ and $\tilde{c}_t$
- Particle filter conditioned on $Z_{t-1}$: SV, $s_t$, . . .

UCSV generates time-varying VAR

Recall:

$$X_t = \begin{bmatrix} S_t \\ M_t \end{bmatrix} \quad Z_t = \begin{bmatrix} Y_t \\ M_t \end{bmatrix}$$
BASELINE FORECAST
Forecast for policy rate, $t = 2015:Q4$: $E(i_{t+h} \mid Z^{t-1})$
UPDATED FORECAST AFTER IMPULSE
After 1pp shadow-rate impulse at $t = 2015:Q4$: $E(i_{t+h} | Z^{t-1}, \varepsilon_t^M)$
POLICY RATE IRF
IRF as change from baseline after 1pp shadow-rate impulse at $t = 2015:Q4$
After 1pp decline in shadow rate, orthogonal to inflation and business cycle.
ACTUAL RATE RESPONSES
After 1pp decline in shadow rate, orthogonal to inflation and business cycle.
**TAKE AWAYS**

**Shadow-rate MP shocks at ELB**

- Less permanent effects on level of interest rates
  
  *During recession, impulses considered largely cyclical*
YIELD SPREAD (10Y ./. 2Y) RESPONSES
After 1pp decline in shadow rate, orthogonal to inflation and business cycle
## TAKE AWAYS

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TAKE AWAYS

Shadow-rate MP shocks at ELB

- Less permanent effects on level of interest rates
  
  *During recession, impulses considered largely cyclical*

- More pronounced effects on term premia
  
  *Picking up unconventional policy?*

- More stimulus of real activity
  
  *Unconventional policy very effective?*
INFLATION RESPONSES
After 1pp decline in shadow rate, orthogonal to inflation and business cycle.
INFLATION GAP RESPONSES
After 1pp decline in shadow rate, orthogonal to inflation and business cycle.
**Shadow-rate MP shocks at ELB**

- Less permanent effects on level of interest rates
  
  *During recession, impulses considered largely cyclical*

- More pronounced effects on term premia
  
  *Picking up unconventional policy?*

- More stimulus of real activity
  
  *Unconventional policy very effective?*

- **Smaller effects on inflation** (short- and long-run)
  
  *Long-run Fisher effects dominates flattening Phillips Curve (conditional on MP shocks)*
CONCLUSIONS

New method

• Shadow-rate sampling extends wide class of “standard” time-series tools to accommodate nominal rates at ELB
• Shadow rate is an unobserved state variable that affects model dynamics and forecasts

Model Results

• Real-rate trend estimates edged down recently, but not significantly so
• Shadow-rate relevant for forecasting nominal rates
• If interpreted as unobserved stance of monetary policy: interesting time-variation in IRFs near the ELB