Longevity risk in annuity portfolios: the effect of product
design and portfolio composition *

Ralph Stevens†  Anja De Waegenaeere‡  Bertrand Melenberg§

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Abstract

The goal of our paper is to identify possibilities for longevity risk management through product design and portfolio composition. We consider two types of defined benefit pension plans that, at retirement age, allow the participant to choose between a single life annuity, and an actuarially equivalent joint and survivor annuity. In one plan, the participant accrues the right to receive a single life annuity, and can exchange that annuity for a joint and survivor annuity. The opposite holds for the other plan. We show that both types of plans are affected by longevity risk in two ways. First, the participants’ choices at retirement age affect product and gender mix, and, therefore, affect the natural hedge potential that arises from combining single life annuities and survivor annuities. Second, uncertainty in the exchange rate induces uncertainty in the level of the nominal rights for old-age pension insurance and partner pension insurance, respectively. We show that both effects have significant impact on longevity risk, and that the effect of exchange rate uncertainty depends strongly on the type of plan.

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†Department of Econometrics and OR, Tilburg University, CentER for Economic Research and Netspar.
‡Corresponding author: Department of Econometrics and OR Tilburg University, CentER for Economic Research and Netspar, PO Box 90153, 5000LE Tilburg, The Netherlands, Phone: +31 13 4662913, Fax: +31 13 4663280, Email: A.M.B.DeWaegenaeere@uvt.nl.
§Department of Econometrics and OR, Department of Finance, Tilburg University, CentER for Economic Research and Netspar.
1 Introduction

Existing literature shows that uncertainty regarding the future development of human life expectancy potentially imposes significant risk on pension funds and insurers (see, e.g., Olivieri and Pitacco 2003; Cossette et al. 2007, and Harí et al. 2008). At present, there is considerable interest in the development of longevity-linked financial instruments such as, for example, longevity bonds. Since the payoff of such securities is linked to the development of survival rates, they can be used to partially hedge longevity risk in pension and life insurance portfolios. There is currently a large body of literature that focusses on the design and valuation of longevity-linked securities (see, for example, Blake et al. 2006, and references therein).

The focus in this paper is on the extent to which the effect of longevity risk on the liabilities of pension funds can be mitigated through product design, in settings where pension funds offer both old-age protection and widowers protection. Many defined benefit pension funds allow participants to choose, at retirement date, between having only old-age pension insurance, or having both old-age pension insurance and partner pension insurance. The former consists of a single life annuity for the life of the participant. The latter consists of a combination of a single life annuity for the life of the participant, and a survivor annuity for the life of the spouse, if (s)he outlives the participant, and is referred to as a joint and survivor annuity. We consider two design characteristics of such optional pension plans. First, there is the ratio of insured rights for the single life annuity over insured rights for the survivor annuity. Second, and more importantly, there is a choice in terms of how pension rights are accrued. We consider two alternatives. First, we consider a defined benefit pension plan in which the participant accrues both old-age pension rights and partner pension rights, i.e. builds up the right to receive a joint and survivor annuity, and, at retirement date, has the option to exchange that annuity for a single life annuity which provides only old-age pension insurance. Second, we consider a defined benefit pension plan in which the participant accrues only old-age pension rights, i.e. builds up the right to receive a single life annuity, and, at retirement date, has the option to exchange that annuity for a joint and survivor annuity.

Our goal in this paper is to investigate the effect of longevity risk on these two types of pension plans. We show that both types of exchange options affect longevity risk in two ways. First, there is an indirect effect due to the fact that the exchange option affects the product mix for both genders, i.e. the ratio of insured rights for single life annuities.

over insured rights for survivor annuities, as well as the gender mix for each product, i.e. the ratio of insured rights for males over insured rights for females. Existing literature shows that the natural hedge potential that arises from combining annuities and death benefits can be substantial (see, e.g., Cox and Lin 2007; Wang et al. 2008; and Tzeng et al. 2008). We show that, although to a lesser extent, survivor annuities also provide a natural hedge for single life annuities. In addition, gender mix affects longevity risk because longevity trends for males and females are not perfectly correlated. Therefore, the participants’ choices at retirement age affect the natural hedge potential that arises from product and gender mix. Second, there is a direct effect of exchange options on longevity risk because, for insureds who did not yet reach retirement age, the rate at which they will be able to exchange one type of annuity for the other type is affected by longevity risk. This occurs because the exchange of annuities occurs at the time they reach retirement age, and has to be actuarially neutral at that time. This exchange rate uncertainty affects both the participant and the fund, because it induces uncertainty in the level of the nominal payments for old-age pension insurance and for partner pension insurance.

The paper is organized as follows. In Section 2, we introduce notation and define two types of pension plans. In Section 3, we model the liabilities of these pension plans. Section 4 deals with the effect of longevity risk. Subsection 4.1 discusses the natural hedge potential that arises from combining old-age and partner pension liabilities, and from combining liabilities of different genders. These results are then used to investigate the indirect effect of exchange options on longevity risk through changes in product and gender mix at retirement age. Subsection 4.2 deals with the effect of exchange rate uncertainty on longevity risk. Subsection 4.4 discusses the difference between individual-specific exchange rates, and gender- and age-neutral exchange rates. Section 5 concludes. All proofs are deferred to Appendix A.

2 Problem definition and notation

Our goal in this paper is to investigate the effect of longevity risk on the liabilities of pension plans that, at retirement age, allow the participant to choose between only old-age pension insurance, or the combination of old-age pension insurance and partner pension insurance. The liabilities of pension plans are affected by two types of longevity risk:

- *micro-longevity risk* because, conditional on given survival probabilities, the remaining lifetime of an individual (participant or partner) is a random variable;
• *macro-longevity risk* because survival probabilities for future dates are random variables.

While micro-longevity risk is diversifiable (i.e. the risk becomes negligible when portfolio size is large), this is not the case for macro-longevity risk. Therefore, our focus in this paper is on the effect of macro-longevity risk.

This section is organized as follows. Subsection 2.1 introduces notation. Subsection 2.2 introduces basic pension annuities, and Subsection 2.3 defines the two types of pension plans discussed in the paper.

### 2.1 Notation

We let survival probabilities depend on age, gender, and calendar year, and use the following notation:

- A participant is characterized by his (her) age $x$, gender $g \in \{m, f\}$, whether (s)he has a partner $p \in \{0, 1\}$, and, if $p = 1$, the age of the partner $y$, and the gender of the partner $g' \in \{m, f\}$.

- $T^{(g)}_{x,t}$ denotes the *remaining lifetime* at time $t$ of an individual with gender $g \in \{m, f\}$ and age $x$ at time $t$. Furthermore, we denote $1_{\{T^{(g)}_{x,t} \geq \tau\}}$ for the indicator random variable that indicates whether an individual (participant or partner) with gender $g$ and age $x$ at time $t$ will survive at least $\tau$ more years, i.e.

$$1_{\{T^{(g)}_{x,t} \geq \tau\}} = \begin{cases} 1 & \text{if } T^{(g)}_{x,t} \geq \tau, \\ 0 & \text{if } T^{(g)}_{x,t} < \tau. \end{cases}$$

- $p^{(g)}_{x,t}$ denotes the probability that an $x$-year-old at date-$t$ with gender $g$ will survive at least another year;

- $\tau p^{(g)}_{x,t} = p^{(g)}_{x,t} \cdot p^{(g)}_{x+1,t+1} \cdots p^{(g)}_{x+\tau-1,t+\tau-1}$ denotes the probability that an $x$-year-old at date-$t$ with gender $g$ will survive at least another $\tau$ years.

- $\mathcal{P}_t = \{p^{(g)}_{x,s} \mid g \in \{m, f\}, x \geq 0, \text{ and } s \leq t\}$ denotes the set of death probabilities for periods $s \leq t$.

The expected present value of liability payments will also depend on the term structure of interest rates. Since our focus in this paper is on longevity risk, we will consider the case where there is no interest rate risk, and let $P^{(\tau)}$ denote the date-$t$ value of one unit to be paid at time $t + \tau$, for every $t$. 

2.2 Basic pension annuities

We consider three types of annuities. In each case, we consider a given date \( t \), and determine the liability payments in all future dates \( t + \tau \), as well as the date-\( t \) expected present value of these payments, conditional on future survival probabilities.\(^2\) The latter is a function of survival probabilities, and is therefore affected by macro-longevity risk.

i) An old-age pension insurance, which consists of a *(deferred) single life annuity*, which yields a nominal yearly payment of 1 in every year that the participant is alive and older than 65. The liability payment in period \( t + \tau \) equals

\[
\tilde{L}_{oa}(x, g, \tau, t) = 1_{\{T(g) \geq \tau\}} 1_{\{x + \tau \geq 65\}}, \text{ for } \tau = 0, 1, \cdots
\]

The date-\( t \) expected present value of future payments, conditional on survival probabilities, equals:

\[
L_{oa}(x, g, t) = \sum_{\tau = \max\{65-x,0\}}^{110-x} \tau p^{(g)}_{x,t} \cdot P^{(\tau)}.
\]

ii) A partner pension insurance, which consists of a *(deferred) survivor annuity*, consisting of a nominal yearly payment of 1 every year that the spouse outlives the participant, in case the insured dies after retirement age. The liability payments, and the corresponding expected present value are given by:\(^3\)

\[
\tilde{L}_{pp}(x, g, g', \tau, t) = 1_{\{65-x \leq T(g) < \tau\}} 1_{\{T(g') \geq \tau\}}, \text{ for } \tau = 0, 1, \cdots
\]

\[
L_{pp}(x, y, g, g', t) = \sum_{\tau = \max\{65-x,0\}}^{110-y} \left( \max\{65-x,0\} p^{(g)}_{x,t} - \tau p^{(g)}_{x,t} \right) \cdot \tau p^{(g')}_{y,t} \cdot P^{(\tau)}.
\]

iii) A temporary partner pension insurance, which consists of a *temporary survivor annuity* consisting of a nominal yearly payment of 1, in every year that the spouse outlives the participant, in case the insured dies before retirement age. The liability payments, and the corresponding expected present value are given

\(^2\)While our focus in this paper is on the effect of longevity risk on the expected present value of liabilities at a given and fixed time, say \( t = 0 \), date-\( t \) expected present values for \( t > 0 \) will be needed to determine exchange rates.

\(^3\)Existing literature shows that there exists dependence between the remaining lifetimes of a participant and his (her) partner at micro-level, e.g. due to the fact that partners have similar lifestyles, or that the passing away of a partner affects the surviving relative’s quality of life. Because our focus in this paper is on macro-longevity risk, we ignore this dependence.
by:

\[
\tilde{L}_{tpp}(x, y, g, g', \tau, t) = \begin{cases} 
1_{(T_{x,t}^{(g)} < \min(65-x, \tau))} 1_{(T_{y,t}^{(g')} > \tau)}, & \text{for } \tau = 0, 1, \cdots \quad (5) 
\end{cases}
\]

\[
L_{tpp}(x, y, g, g', t) = \sum_{\tau=1}^{110-y} \left(1 - \min(65-x, \tau) p_{x,t}^{(g)}\right) \tau p_{y,t}^{(g')} \cdot P^{(\tau)}. \quad (6)
\]

### 2.3 Pension plan definition

We consider two pension plans that, at retirement age, allow the participant to choose between only old-age pension insurance, i.e. a single life annuity, or the combination of old-age pension insurance and partner pension insurance, i.e. a single life annuity combined with a survivor annuity. The two plans differ with respect to the accrual of pension rights. In the first plan, which we refer to as a ‘JointLife plan’, the participant accrues both old-age and partner pension rights. At retirement date, the insured has the option to exchange the partner pension rights for additional old-age pension rights. In the second plan, which we refer to as a ‘SingleLife plan’, the participant builds up only old-age pension rights, and, at retirement date, has the option to exchange some old-age pension rights for partner pension rights. Formally, the two plans are defined as follows:

- In a JointLife plan, the participant accrues the right to receive a deferred single life annuity with nominal payment of 1, combined with a deferred survivor annuity with nominal payment of \(w\), with the option to exchange, at retirement age and at an actuarially neutral rate, the survivor annuity for additional single life annuity;

- In a SingleLife plan, the participant accrues the right to receive a single life annuity with nominal payment of 1, with the option to exchange, at retirement age and at an actuarially neutral rate, part of the single life annuity for a survivor annuity. After exchange, the ratio of the nominal payment of the survivor annuity over the nominal payment of the single life annuity should be equal to \(w\).

In both cases, the policy may include temporary partner pension insurance in the form of a temporary survivor annuity, which provides a partner pension in case of decease of the insured before retirement age.

### 3 Pension plan liabilities

In this section, we model the liabilities of the two types of pension plans introduced in Subsection 2.3. For both types of pension plan, the liabilities consist of a combination
of single life annuity and survivor annuity, but the nominal rights depend on the type of pension plan, the rate at which participants will be able to exchange old-age pension rights for partner pension rights, or vice versa, and the participants’ choices at retirement date.

This section is therefore organized as follows. Subsection 3.1 determines the actuarially neutral exchange rate for the exchange of partner pension rights for additional old-age pension rights, or vice versa. Subsection 3.2 models the liability payments as a function of the participant’s choice at retirement date. Subsection 3.3 models the participant’s choice at retirement date. Finally, Subsection 3.4 combines the result of the previous three subsections to determine the expected present value of the liability payments in the two types of plans.

Throughout this section, we consider an insured who is alive, currently (i.e. at time $t = 0$) aged $x$, with gender $g$, and, in case $p = 1$, with a partner with gender $g'$ currently aged $y$.

### 3.1 The exchange rate

The *JointLife* plan includes the option to exchange partner pension rights for old-age pension rights. In contrast, the *SingleLife* plan includes the option to exchange old-age pension rights for partner pension rights. The liabilities of these pension plans therefore depend on the rate at which a single life annuity can be exchanged for a survivor annuity, and vice versa.

Pension laws typically prescribe that the exchange of one type of liability for another type of liability should be actuarially neutral. This implies that the exchange rate should be such that the expected present value of the liability payments after exchange equals the expected present value of the liability payments before exchange. While gender "discrimination" is typically not allowed by law (i.e. the exchange rate cannot depend on the gender of the insured and his (her) partner), pension laws vary in terms of how gender neutral liability values should be determined. Some countries allow the exchange rate to depend on the age of the partner (such as, for example, the United States); others prohibit such age "discrimination" (for example, the Netherlands).

In this paper, we consider a gender- and age-neutral exchange rate, where we let gender- and age-neutral liability value be defined as the average of the expected present value, with respect to gender neutral survival probabilities, of the liability payments for an insured with a three year younger partner, and an insured with a three year older

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4Defined benefit plans in the US are tax favored only if gender neutral values are determined on the basis of unisex life tables (see IRC section 417(e)(3)).
partner, i.e.\(^5\)

\[
L_{oa,n}(t) = \mathbb{E}[L_{oa}(65, n, t) | \mathcal{P}_t],
\]

\[
L_{pp,n}(t) = \frac{1}{2}\mathbb{E}[L_{pp}(65, 62, n, n, t) | \mathcal{P}_t] + \frac{1}{2}\mathbb{E}[L_{pp}(65, 68, n, n, t) | \mathcal{P}_t],
\]

where \(L_{oa}(65, n, t), L_{pp}(65, 62, n, n, t), \) and \(L_{pp}(65, 68, n, n, t)\) are given by (2), and (4) with \(\tau_{p(x,t)}^{(g)}\) and \(\tau_{p(y,t)}^{(g')}\) replaced by gender neutral values defined as:

\[
\tau_{p(x,t)}^{(n)} = \frac{1}{2}\left(\tau_{p(x,t)}^{(m)} + \tau_{p(x,t)}^{(f)}\right), \text{ for all } x, t, \tau.
\]

Then, in a JointLife plan, actuarially neutral exchange, at date \(t\), of a nominal yearly partner pension right of 1 yields an (extra) nominal yearly old-age pension right of \(e(t)\), where

\[
e(t) = \frac{L_{pp,n}(t)}{L_{oa,n}(t}).
\]

In a SingleLife plan, actuarially neutral exchange at date \(t\) of a nominal yearly old-age pension right of 1 yields a nominal yearly partner pension right of \(\frac{1}{e(t)}\).

Note that an insured currently (i.e. at date \(t = 0\)) aged \(x < 65\) will have the option to exchange pension rights at retirement age, i.e. at date \(t = 65 - x\). Therefore, for all insureds who are currently younger than 65, the exchange rate \(e(65 - x)\) is a random variable, and is affected by macro-longevity risk, because it depends on \(\mathcal{P}_{65-x}\), the set of death probabilities between now and \(t = 65 - x\).\(^6\)

### 3.2 Liability payments

In this subsection, we determine the liability payments at any given future date \(\tau\) for the two types of pension plans introduced in Subsection 2.3. To avoid overloaded notation, we will use the following simplified notation:

\[
e := e(65 - x) \quad \wedge \quad \tilde{L}_{oa}(\tau) := \tilde{L}_{oa}(x, g, \tau, 0),
\]

\[
\wedge \quad \tilde{L}_{pp}(\tau) := \tilde{L}_{pp}(x, y, g', \tau, 0),
\]

\[
\wedge \quad \tilde{L}_{tp}(\tau) := \tilde{L}_{tp}(x, y, g', \tau, 0).
\]

\(^5\)The age difference is based on the average age difference in married couples (see, e.g. Brown and Poterba 2000). In Subsection 4.4, we will discuss alternative exchange rate definitions.

\(^6\)The exchange rate is affected by interest rate risk, because it depends on the term structure of interest rates at time \(t = 65 - x\). Because our focus in this paper is on longevity risk, we ignore interest rate risk.
First, consider a date \( \tau < 65 - x \). Then, the participant has not yet reached retirement age. Therefore, for both types of pension plans, the liability payment \( \tilde{L}(\tau) \) consists either of a temporary partner pension payment, or no payment, depending on survival of the insured, whether there is a partner, and, if so, on survival of the partner. Since the nominal temporary partner pension right equals \( w \), and since for a participant who does not have a partner \( (p = 0) \), there are no temporary partner pension payments, it holds that

\[
\begin{align*}
\tilde{L}(\tau) &= w \cdot \tilde{L}_{tpp}(\tau), & \text{for } p = 1, \tau < 65 - x, \\
&= 0, & \text{for } p = 0, \tau < 65 - x.
\end{align*}
\] (11)

Now consider a future date \( \tau \geq 65 - x \). The liability payment then is either an old-age pension payment, a (temporary) partner pension payment, or no payment, depending on survival of the insured and the partner, if there is one \( (p = 1) \). The liability therefore consists of a combination of a single life annuity and a survivor annuity. The nominal payment for the single life annuity \( (A_{oa}) \), and the nominal payment for the survivor annuity \( (A_{pp}) \), depend on the type of pension plan, the exchange rate \( e(65 - x) \), and the insured’s choice at retirement date. For the latter, we let

\[
1_{OA} := 1_{OA}(x, y, g, g') \in \{0, 1\},
\] (12)

be a random variable that, conditional on the insured being alive at time \( t = 65 - x \), equals 1 if he will choose to hold only old-age pension insurance, and 0 otherwise.

First, consider an insured in a JointLife plan. If the insured at time \( t = 65 - x \) will prefer to hold both old-age insurance and partner pension insurance \( (1_{OA} = 0) \), the nominal payment for the single life annuity equals \( A_{oa} = 1 \), and the nominal payment for the survivor annuity equals \( A_{pp} = w \). If the insured will prefer to hold only old-age pension insurance \( (1_{OA} = 1) \) (e.g., because he no longer has a partner), he exchanges partner pension rights for old-age pension rights. Since partner pension rights consist of a nominal yearly payment of \( w \), actuarially neutral exchange of those rights yields additional old-age pension rights of \( w \cdot e \), where the exchange rate \( e \) is as defined in (10). Therefore, the nominal yearly payment of the single life annuity equals \( A_{oa} = 1 + w \cdot e \), and there is no survivor annuity, so \( A_{pp} = 0 \). Thus,

\[
(A_{oa}, A_{pp}) = \begin{cases} 
(1, w), & \text{if } 1_{OA} = 0, \\
(1 + w \cdot e, 0), & \text{if } 1_{OA} = 1.
\end{cases}
\] (13)

Since for a participant who does not have a partner \( (p = 0) \), there are no partner pension
payments, (11) and (13) imply that:

\[ \tilde{L}(\tau) = \begin{cases} (1 + w \cdot e \cdot 1_{OA}) \cdot \tilde{L}_{oa}(\tau), & \text{for } p = 0; \\ w \cdot \tilde{L}_{pp}(\tau), & \text{for } p = 1, \tau < 65 - x; \\ (1 + w \cdot e \cdot 1_{OA}) \cdot \tilde{L}_{oa}(\tau) + w \cdot (1 - 1_{OA}) \cdot \tilde{L}_{pp}(\tau) + w \cdot \tilde{L}_{pp}(\tau), & \text{for } p = 1, \tau \geq 65 - x. \end{cases} \]

(14)

Next, consider an insured in a SingleLife plan. If the insured chooses to hold only old-age insurance ($1_{OA} = 1$), the nominal yearly payment for the single life annuity equals $A_{oa} = 1$, and there is no survivor annuity, so $A_{pp} = 0$. If the insured prefers to hold both old-age insurance and partner pension insurance ($1_{OA} = 0$), he exchanges some old-age pension rights for partner pension rights. The plan specifies that the ratio of partner pension rights over old-age pension rights after exchange has to be equal to $w$. This implies that $\hat{e}$ of old-age pension rights are exchanged for $w \cdot (1 - \hat{e})$ of partner pension rights, where actuarial neutrality implies that:

\[ L_{oa,n} = L_{oa,n} - \hat{e} \cdot L_{oa,n} + w \cdot (1 - \hat{e}) \cdot L_{pp,n}, \]

or, equivalently,

\[ \hat{e} = w \frac{L_{pp,n}}{L_{oa,n} + w \cdot L_{pp,n}} = \frac{w \cdot e}{1 + w \cdot e}. \]

Therefore, for an insured in a SingleLife plan who chooses to hold both old-age insurance and partner insurance, the nominal yearly payment for the single life annuity equals $A_{oa} = 1 - \hat{e} = \frac{1}{1 + w \cdot e}$, and the nominal yearly payment for the survivor annuity equals $A_{pp} = w(1 - \hat{e}) = \frac{w}{1 + w \cdot e}$. Therefore,

\[ (A_{oa}, A_{pp}) = \begin{cases} \frac{1}{1 + w \cdot e}(1, w), & \text{if } 1_{OA} = 0, \\ \frac{1}{1 + w \cdot e}(1 + w \cdot e, 0), & \text{if } 1_{OA} = 1. \end{cases} \]

(15)

Combined with (11), this yields:

\[ \tilde{L}(\tau) = \begin{cases} \frac{1 + w \cdot e \cdot 1_{OA}}{1 + w \cdot e} \cdot \tilde{L}_{oa}(\tau), & \text{for } p = 0; \\ w \cdot \tilde{L}_{pp}(\tau), & \text{for } p = 1, \tau < 65 - x; \\ \frac{1 + w \cdot e \cdot 1_{OA}}{1 + w \cdot e} \cdot \tilde{L}_{oa}(\tau) + \frac{w(1 - 1_{OA})}{1 + w \cdot e} \cdot \tilde{L}_{pp}(\tau) + w \cdot \tilde{L}_{pp}(\tau), & \text{for } p = 1, \tau \geq 65 - x. \end{cases} \]

(16)
3.3 The choice at retirement age

Let us now turn to choice behavior. It follows from (13) and (15) that, for both types of plans, the insured has the option to choose between two combinations of a single life annuity and a survivor annuity with insured rights given by

\[(A_{oa}, A_{pp}) = c \cdot (1, w), \text{ and } (A_{oa}, A_{pp}) = c \cdot (1 + w \cdot e, 0),\]  

(17)

respectively, where \(c = 1\) in case of the JointLife plan, and \(c = \frac{1}{1+w \cdot e}\) in case of a SingleLife plan.\(^7\) To specify the probability distribution of the choice indicator \(1_{OA}\) defined in (12), we distinguish three types of insureds.

i) For an insured currently aged \(x < 65\) who does not have a partner (i.e. \(p = 0\)), partner pension insurance has no value, and so he will prefer to hold only old-age pension insurance. Therefore, \(1_{OA} = 1\);

ii) For an insured currently aged \(x < 65\) who does have a partner (i.e. \(p = 1\)), the choice will depend on whether the partner is still alive at date \(t = 65 - x\). If the partner is no longer alive, partner pension insurance has no value, and the insured will prefer to hold only old-age insurance, and so \(1_{OA} = 1\). If the partner is still alive at time \(t = 65 - x\), partner pension insurance does have value. The choice between only old-age insurance and combined old-age insurance and partner pension insurance will then depend on the couple’s preference relation between a single life annuity and a joint and survivor annuity with nominal payments as given in (17) (see, e.g., Johnson, Uccello and Goldwyn 2005). We denote \(\alpha = \alpha(x, y, g, g')\) for the probability that the couple, conditional on both being alive at time \(t = 65 - x\), will prefer to hold both old-age insurance and partner pension insurance.\(^8\) Then,

\[1_{OA} = \left(1 - \frac{T_{g',0}^{(g')} < 65 - x}{1 - F_{y,0}^{(g')} < 65 - x}} \right) \left(1 - F_{x,0}^{(g')} \right) \]

\[= \left(1 - F_{x,0}^{(g')} \right), \quad (18)\]

where \(F\) is a random variable that, conditional on \(P_{\infty}\), is independent of \(T_{x,0}^{(g')}\) and

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\(^7\)Note that for insureds with CRRA utility functions, the choice is independent of the nominal rights, and (17) implies that the choice will be the same in the two types of pension plans.

\(^8\)The probability \(\alpha\) is likely to depend on the ratio of insured rights for partner pension over insured rights for old-age pension, as well as the nominal insured right for old-age pension (see, e.g. Brown and Poterba 2000). In our model, the former equals \(w\), and the latter is normalized to 1.
\[ T_{y,0}^{(g')}, \text{ and} \]
\[ P(F = 1 | P_{65-x}) = \alpha, \]
\[ P(F = 0 | P_{65-x}) = 1 - \alpha; \tag{19} \]

iii) For an insured currently aged \( x \geq 65 \), the choice has already been observed, and \( 1_{OA} = 1 \) (\( 1_{OA} = 0 \)) if the insured preferred to hold only old-age pension insurance (preferred to hold both old-age pension insurance and partner pension insurance).

Note that for an insured currently aged \( x < 65 \) who has a partner, the choice indicator \( 1_{OA} \) is a random variable that is affected by both micro- and macro longevity risk, since it depends on survival of the spouse, \( (T_{y,0}^{(g')}) \geq 65 - x \), and, conditional on survival of the spouse, it depends on the preference ordering, at date \( t = 65 - x \), between the two types of annuities. The latter is likely to depend on \( e \), and may depend on survival rates of the two spouses at the time the decision is made. Therefore, we allow the probability \( \alpha \) to depend on \( P_{65-x} \), i.e. the realization of death probabilities between now and time \( t = 65 - x \).

### 3.4 Expected present value of future liabilities

Again consider an insured currently aged \( x \) with gender \( g \), with, in case \( p = 1 \), a partner with gender \( g' \) currently aged \( y \), and let \( L \) denote the expected present value of future payments, conditional on future death probabilities, for that participant in case of a JointLife plan or a SingleLife plan, i.e.

\[ L = \sum_{\tau = 0}^{110-x} \mathbb{E} \left[ \tilde{L}(\tau) | P_{\infty} \right] P^{(\tau)}, \tag{20} \]

where \( \tilde{L}(\tau) \) is given by (14) in case of a JointLife plan, and by (16) in case of SingleLife plan.

In this section, we express \( L \) as a linear combination of \( L_{oa} \), \( L_{pp} \), and \( L_{tpp} \), where

\[ L_{oa} := L_{oa}(x, g, 0), \]
\[ L_{pp} := L_{pp}(x, y, g, g', 0), \]
\[ L_{tpp} := L_{tpp}(x, y, g, g', 0), \]

denote the expected present value of future payments, conditional on \( P_{\infty} \), for the single life annuity, the survivor annuity, and the temporary survivor annuity, as defined in (2),
For an insured who already reached retirement age, i.e. \( x \geq 65 \), the exchange rate \( e \), as well as the choice indicator \( 1_{OA} \) are known. In addition, there are no temporary partner pension payments. Therefore, the following proposition follows immediately from (14), (16), and (20).

**Proposition 1** Consider an insured with age \( x \geq 65 \). Then,

i) if the insured has a partner, \( L = \delta_1 \cdot L_{oa} + \delta_2 \cdot L_{pp} \), where, for the JointLife plan,

\[
\delta_1 = 1 + w \cdot e \cdot 1_{OA}, \\
\delta_2 = w \cdot (1 - 1_{OA}),
\]

and, for the SingleLife plan,

\[
\delta_1 = \frac{1}{1 + w \cdot e} \left(1 + w \cdot e \cdot 1_{OA}\right), \\
\delta_2 = \frac{w}{1 + w \cdot e} \cdot (1 - 1_{OA});
\]

ii) if the insured does not have a partner, \( L = (1 + w \cdot e \cdot 1_{OA}) \cdot L_{oa} \), for the JointLife plan, and \( L = \frac{1}{1 + w \cdot e} (1 + w \cdot e \cdot 1_{OA}) \cdot L_{oa} \) for the SingleLife plan.

For insureds who did not yet reach retirement age, i.e. \( x < 65 \), there is uncertainty with regard to their choice between a single life annuity and a joint life annuity at date \( t = 65 - x \). In addition, the value of the exchange rate \( e \) is uncertain as it depends on the development of death rates until retirement age. Moreover, for participants with a partner (i.e. \( p = 1 \)), the value of partner pension liabilities and the choice indicator random variable are not independent at micro-level, because they both depend on survival of the partner (i.e. \( T^{(q')}_{y,0} \)). This yields the following result.

**Proposition 2** Consider an insured with age \( x < 65 \). Then,

i) if the insured has a partner, \( L = \delta_1 \cdot L_{oa} + \delta_2 \cdot L_{pp} + w \cdot L_{tp} \), where, for the JointLife plan,

\[
\delta_1 = 1 + \left(1 - \alpha \cdot 65 - x \cdot P^{(q')}_{y,0} \right) \cdot w \cdot e; \\
\delta_2 = \alpha \cdot w;
\]

\[\text{(21)}\]
and, for the SingleLife plan,

\[
\begin{align*}
\delta_1 &= \frac{1}{1 + w \cdot e} \left( 1 + \left( 1 - \alpha \cdot 65-x_p^{(g')}(g) \cdot w \cdot e \right) \right); \\
\delta_2 &= \frac{1}{1 + w \cdot e} \cdot \alpha \cdot w.
\end{align*}
\] (22)

ii) If the insured does not have a partner, \( L = (1 + w \cdot e) \cdot L_{oa} \), for the JointLife plan, and \( L = L_{oa} \) for the SingleLife plan.

4 Longevity risk

It follows from Proposition 1 and Proposition 2 that exchange options affect longevity risk in two ways. First, the participants’ choices at retirement age may affect the product mix (ratio of survivor annuity rights over single life annuity rights) for both genders, as well as the gender mix for each type of annuity (ratio of male rights over female rights). Second, for insureds who are not yet retired (i.e. \( x < 65 \)), the exchange rate \( e \) is a random variable.

In section 4.1 we investigate the indirect effect of exchange options on longevity risk through changes in product and gender mix at age 65. In section 4.2, we investigate the direct effect of exchange options on longevity risk due to uncertainty in the exchange rate for younger insureds. In each case, it is assumed that:

- the male insured has a partner who is three years younger (\( y = x - 3 \)), and the female insured has a partner who is three years older (\( y = x + 3 \));
- partners are of different gender, i.e. \( g = m \) implies \( g' = f \), and vice versa;
- the term structure of interest rates is flat at 4%, i.e., \( P^{(r)} := 1/(1 + 0.04)^r \).

In order to quantify longevity risk, we use stochastic forecast models to forecast the probability distribution of future death probabilities. In all our calculations, we include three sources of risk: process risk, parameter risk, and model risk. For a detailed description of the models and estimation techniques, we refer to Appendix C.

4.1 Effect of product and gender mix changes at age 65

Since for 65 year old insureds, there is no longer a temporary partner pension liability, there are four different liabilities: old-age pension insurance (i.e. a single life annuity) for a 65 year old male, respectively female, and, partner pension insurance (i.e. a survivor
annuity) for a 65 year old male insured with a female partner aged $y(m) = 62$, and for a 65 year old female insured with a male partner aged $y(f) = 68$, respectively. We denote

$$L_{oa,g} := L_{oa}(65, g, 0), \quad (23)$$
$$L_{pp,g} := L_{pp}(65, y(g), g, g', 0), \quad (24)$$

for $g \in \{m, f\}$, for the corresponding expected present value of the liability payments, conditional on future survival probabilities, as defined in (2) and (4), respectively. In addition, we let

- $g(i) \in \{m, f\}$ denote the gender of participant $i$;
- $M = \{i : g(i) = m\}$ denote the set of male participants, and we let $F = \{i : g(i) = f\}$ denote the set of female participants.
- $C(i)$ denote the insured right for old-age pension payments of participant $i$, before exchange;
- $\gamma$ denote the fraction of male insured rights before exchange, i.e.
  $$\gamma = \frac{\sum_{i \in M} C_i}{\sum_i C_i};$$
- $1_{OA}(i) = 1$ if insured $i$ chose to only hold old-age pension insurance, and $1_{OA}(i) = 0$ otherwise;
- $\tilde{\alpha}_m$ denote the fraction of male rights for old-age insurance for male participants who chose to hold both old-age pension insurance and partner pension insurance; and let $\tilde{\alpha}_f$ denote the female counterpart, i.e.
  $$\tilde{\alpha}_m = \frac{\sum_{i \in M} C_i \cdot (1 - 1_{OA}(i))}{\sum_{i \in M} C_i}, \quad \text{and} \quad \tilde{\alpha}_f = \frac{\sum_{i \in F} C_i \cdot (1 - 1_{OA}(i))}{\sum_{i \in F} C_i},$$

Now let us denote $L_{JointLife}(L_{SingleLife})$ for the expected present value of the aggregate liabilities at age 65, conditional on $\mathcal{P}_\infty$, in case of a JointLife (SingleLife) plan. Then, we have the following result.

**Proposition 3** For any given $\gamma$, $\tilde{\alpha}_m$, and $\tilde{\alpha}_f$, it holds that:

$$\frac{\sigma(L_{JointLife})}{\mathbb{E}[L_{JointLife}]} = \frac{\sigma(L_{SingleLife})}{\mathbb{E}[L_{SingleLife}]} = \frac{\sigma(L)}{\mathbb{E}[L]},$$
where

\[ L = \tilde{\gamma} \cdot (L_{oa,m} + \tilde{w}_m \cdot L_{pp,m}) + (1 - \tilde{\gamma}) \cdot (L_{oa,f} + \tilde{w}_f \cdot L_{pp,f}) \]  

(25)

with

\[ \tilde{\gamma} = \frac{\gamma \cdot (1 + (1 - \tilde{\alpha}_m) \cdot w \cdot e)}{\gamma \cdot (1 + (1 - \tilde{\alpha}_m) \cdot w \cdot e) + (1 - \gamma) \cdot (1 + (1 - \tilde{\alpha}_f) \cdot w \cdot e)}, \]  

(26)

\[ \tilde{w}_g = \frac{\tilde{\alpha}_g \cdot w}{(1 + (1 - \tilde{\alpha}_g) \cdot w \cdot e)} \in [0, w], \text{ for } g \in \{m, f\}. \]  

(27)

The above proposition shows that the liabilities can be seen as a weighted average of joint and survivor annuities for males and females, where the nominal right for the survivor annuity for gender \( g \) is given by \( \tilde{w}_g \). The gender mix \( \tilde{\gamma} \), and the product mix for males and females, \( \tilde{w}_m \) and \( \tilde{w}_f \), are functions of the participants’ choices between a single life annuity and a joint and survivor annuity.

The extent to which longevity risk in the portfolio will be affected by the participants’ choices therefore depends on the extent to which there is hedge potential from combining different types of liabilities (i.e. single life annuity and survivor annuity) for a given gender, or the same type of liability for different genders. This, in turn, depends on the correlation between the four types of liabilities. Let \( \Sigma \in \mathbb{R}^{4 \times 4} \) denote the correlation matrix of the vector \((L_{oa,m}, L_{oa,f}, L_{pp,m}, L_{pp,f})\). Using the simulation models described in the appendix combined with expressions (2) and (4) yields:

\[
\Sigma = \begin{pmatrix}
L_{oa,m} & L_{oa,f} & L_{pp,m} & L_{pp,f} \\
L_{oa,m} & 1 & -0.60 & 0.32 \\
L_{oa,f} & 0.27 & 1 & -0.78 \\
L_{pp,m} & -0.60 & 0.60 & 1 \\
L_{pp,f} & 0.32 & -0.78 & -0.86 & 1
\end{pmatrix}.
\]  

(28)

We observe the following:

i) The correlation between single life annuities for males and females \((L_{oa,m} \text{ and } L_{oa,f})\) is relatively low. Although the correlation in the trend forecasts is high, the correlation in the parameter estimates for males and females is low.

ii) The correlation between survivor annuities for males and females \((L_{pp,m} \text{ and } L_{pp,f})\) is strongly negative.

iii) For both genders, the correlation between a single life annuity and a survivor annuity \((L_{oa,g} \text{ and } L_{pp,g})\) is negative. A higher realization of old-age pension
liabilities (due to lower mortality rates of the insured) leads to lower partner benefits due to delayed partner pension payments.

These observations indicate that hedge potential of gender and product mix may be substantial. To investigate this issue, we consider the effect of gender and product mix by quantifying $\sigma(L)/E[L]$ as a function of $\tilde{\gamma}$, $\tilde{w}_m$, and $\tilde{w}_f$, for $L$ as defined in (25).

Figure 1 illustrates the the effect of product and gender mix on longevity risk, when the product mix is equal for both genders, i.e. $\tilde{w}_m = \tilde{w}_f = \tilde{w}$. The left panel illustrates the effect of gender mix ($\tilde{\gamma}$) on longevity risk in single life annuities (solid line), in joint and survivor annuities with $\tilde{w} = 0.5$ (dotted line), and in joint and survivor annuities with $\tilde{w} = 1$ (dashed line). The right panel illustrates the effect of product mix on longevity risk in male funds ($\tilde{\gamma} = 1$; solid line), female funds ($\tilde{\gamma} = 0$; dashed line), and funds with 50 percent male rights and 50 percent female rights ($\tilde{\gamma} = 0.5$; dashed-dotted line).

We observe the following:

i) **Gender mix affects longevity risk in single life annuities as well as in joint and sur-**
vivor annuities. This occurs because the death probabilities of males and females are not perfectly correlated.

ii) For joint and survivor annuities with $\tilde{w} = 1$, the effect of gender mix on longevity risk in joint and survivor annuities is small. This occurs because with $\tilde{w} = 1$, the joint and survivor annuity is equivalent to a joint life annuity for the longest survivor, regardless of whether that is the insured or the partner. If both male and female insured would have partners of equal age, the two liabilities would be identical, so that gender composition would have no effect on longevity risk. However, because of the age difference between partners of a male and female insured, respectively, there is a marginal effect of gender composition.

iii) For every gender mix $\tilde{\gamma}$, there is hedge potential from combining single life annuities and survivor annuities. This occurs because, for both men and women, the correlation between the single life annuity and the survivor annuity is negative (-0.60 and -0.78, respectively). Therefore, joint and survivor annuities are less sensitive to longevity risk than single life annuities.

Because the hedge potential of gender and product mix is significant, longevity risk may be substantially affected by the insureds’ choices with regard to whether or not they want to hold partner pension insurance. Proposition 3 shows how the participants’ choices (i.e. $\tilde{\alpha}_m$ and $\tilde{\alpha}_f$) affect the gender mix ($\tilde{\gamma}$), and the product mix for each gender ($\tilde{w}_m$ and $\tilde{w}_f$). Specifically, it follows from (26) and (27) that:

i) for each gender $g \in \{m, f\}$, the ratio of insured rights for survivor annuities over insured rights for single life annuities is increasing in $\tilde{\alpha}_g$;

ii) the fraction of male rights in the portfolio of single life annuities is decreasing in $\tilde{\alpha}_m$ and increasing in $\tilde{\alpha}_f$;

As long as insureds of different genders have the same preferences, i.e. $\tilde{\alpha}_m = \tilde{\alpha}_f = \tilde{\alpha}$, the product mix is equal for the two genders, i.e. $\tilde{w}_m = \tilde{w}_f = \tilde{w}$. A higher preference for joint and survivor annuities (i.e. a higher $\tilde{\alpha}$) will lead to a higher $\tilde{w}$, but it will not affect the gender mix $\tilde{\gamma}$. However, males and females may be significantly different with regard to their preferences between a single life and a joint and survivor annuity. Then, also product mix may be affected by changes in preferences. The left panel of Figure 2 displays $\sigma(L)/E[L]$ as a function of male preference, $\tilde{\alpha}_m$, for selected values

---

\*Johnson, Uccello and Goldwyn (2005) find that 28% of married men and 67% of married women prefer single life annuities.
of female preferences, $\tilde{\alpha}_f$. The right panel displays $\sigma(L)/E[L]$ as a function of female preference, for selected values of male preferences. In each case, the nominal payment for the survivor annuity equals $w = 2/3$.

We observe the following:

i) For both genders, an increase in $\tilde{\alpha}_g$ decreases longevity risk more when $\tilde{\alpha}_g'$ is higher. This occurs because there is a strong negative correlation between partner pension liabilities of different genders (see (28)).

ii) The effect of the participants’ choices on longevity risk due to changes in the product mix ($\tilde{w}_m$ and $\tilde{w}_f$) is larger than the effect of changes in the gender mix ($\tilde{\gamma}$).

### 4.2 Effect of exchange rate risk

In this section we investigate the direct effect of exchange options on longevity risk due to the fact that, for insureds who did not yet reach retirement age, the actuarially neutral exchange rate is a random variable that depends on the realization of death probabilities between now and time at which the insured will reach retirement age. This uncertainty
affects both the participant and the fund. It affects the participant because, depending on the preferences between a single life annuity and a joint and survivor annuity, and depending on the type of pension plan, exchange rate uncertainty induces uncertainty in the level of the nominal payments. It affects the pension fund because the uncertainty in the level of nominal insured rights induces uncertainty in the product and gender mix after retirement.

In this section we consider a given insured aged $x$ at time $t = 0$, with gender $g$, and, in case $p = 1$, with a partner with gender $g'$ currently aged $y$, and investigate the consequences of exchange rate uncertainty for both the participant and the insurer. We let $e := e(65 - x)$.

### 4.2.1 Effect on the participant

First, we consider the effect of exchange rate uncertainty on the participant. While the longevity risk due to uncertainty in the duration of the payments is borne by the pension fund, the participant also bears risk because, depending on his choice between a single life annuity and a joint and survivor annuity, the level of the nominal payments may depend on the exchange rate. Specifically, recall that, in case of a JointLife plan, the nominal payments for old-age pension insurance and partner pension insurance are given by:

$$
(A_{oa}, A_{pp}) = (1, w), \quad \text{if } 1_{OA} = 0,
= (1 + w \cdot e, 0), \quad \text{if } 1_{OA} = 1.
$$

In case of a SingleLife plan, they are given by:

$$
(A_{oa}, A_{pp}) = \left( \frac{1}{1 + w \cdot e}, \frac{w}{1 + w \cdot e} \right), \quad \text{if } 1_{OA} = 0,
= (1, 0), \quad \text{if } 1_{OA} = 1.
$$

Therefore, in case of a JointLife plan, uncertainty in the exchange rate ($e$) affects the level of the nominal payments for old-age pension insurance in case the insured will prefer a single life annuity ($1_{OA} = 1$). In case of a SingleLife plan, uncertainty in the exchange rate induces uncertainty in both the level of the nominal old-age pension as well as the partner pension in case the insured will prefer to hold a joint and survivor annuity ($1_{OA} = 0$). Figure 3 displays selected quantiles of $e(65 - x)$ as a function of $x$. We observe the following:

i) The probability distribution of the exchange rate shifts downwards for younger cohorts.
ii) For young insureds, the uncertainty in the exchange rate is substantial.

This implies that, compared to a current retiree, a younger insured in a *SingleLife* plan who will prefer to hold both old-age pension insurance and partner pension insurance will, with high probability, have to give up less old-age pension rights \((1 - \frac{1}{1 + w \cdot e}) \frac{w}{1 + w \cdot e}\) and receive more partner pension rights \(\frac{w}{1 + w \cdot e}\). However, there is substantial uncertainty regarding the level of both old-age pension rights and partner pension rights. Whereas, when \(w = 2/3\), a current retiree would have to give up 8.5\% of old-age pension rights to receive partner pension rights, the 95\% confidence interval for a 25 year old is [6\%, 8\%].

In contrast, an insured in a *JointLife* plan who will prefer to hold only old-age pension insurance will, with high probability, receive less old-age pension rights \((w \cdot e)\) in exchange for partner pension rights \((w)\). Whereas, when \(w = 2/3\), a current retiree would receive an increase of 14\% in old-age pension rights in return for partner pension rights, the 95\% confidence interval for a 25 year old is [9.5\%, 13\%].

4.2.2 Effect on the pension fund

Next, we consider the effect of exchange rate uncertainty on the expected present value of the liabilities. Recall from Proposition 2 that, for both types of pension plans, the
The expected present value of future payments is of the form

\[ L = \delta_1 \cdot L_{oa} + \delta_2 \cdot L_{pp} + \delta_3 \cdot L_{tpp}, \]

where \( \delta_3 \) is zero in case the insured does not have a partner \( (p = 0) \), and \( w \) otherwise, and is therefore independent of the exchange rate. In contrast, \( \delta_1 \) and \( \delta_2 \) are functions of the exchange rate. Therefore, \( L \) is affected by exchange rate uncertainty in two ways:

i) Since the nominal rights for old-age pension insurance and/or partner pension insurance, \( \delta_1 \) and \( \delta_2 \), respectively, are functions of the exchange rate (see Proposition 2), exchange rate uncertainty implies that \( \delta_1 \) and \( \delta_2 \) are uncertain. This implies that the product and gender mix after retirement is uncertain.

ii) Second, since \( e, L_{oa} \), and \( L_{pp} \) all depend on future death probabilities (\( e \) depends on \( \mathcal{P}_{65-x}; L_{oa} \), and \( L_{pp} \) depend on \( \mathcal{P}_\infty \)), longevity risk induces correlation between \( e, L_{oa} \), and \( L_{pp} \). Since \( \delta_1 \) and \( \delta_2 \) are functions of the exchange rate, this implies that exchange rate uncertainty induces correlation between the level of the payments \( \delta_1 \) and \( \delta_2 \), and the “duration” of the payments \( L_{oa} \) and \( L_{pp} \), respectively.

To understand the importance of these effects, we first investigate the correlations between the exchange rate and the duration of payments, as a function of the age \( x \) of the participant.

Figure 4 displays the correlation between \( e \) and \( L_{oa} \) (dashed lines) for males (lower graph) and for females (upper graph), as well as the correlation between \( e \) and \( L_{pp} \) (dotted lines) for males (upper graph) and for females (lower graph), as a function of \( x \).

We observe the following:

i) For all \( x < 65 \), and for both genders, the correlation between \( e \) and \( L_{oa} \) is negative, and the correlation between \( e \) and \( L_{pp} \) is positive.\(^{10}\) This occurs because

\[ e = \frac{L_{pp,n}}{L_{oa,n}}, \]

and \( L_{oa,n} \) and \( L_{oa} \), as well as \( L_{pp,n} \) and \( L_{pp} \), are positively correlated. In contrast, \( L_{oa,n} \) and \( L_{oa} \), as well as \( L_{pp,n} \) and \( L_{oa} \), are negatively correlated.

\(^{10}\)For \( x > 60 \) the \( \rho(e, L_{pp}) \) is positive for women, which is due to the strong negative correlation between partner pension liabilities for men and women and a low correlation between \( L_{pp} \) for women and \( L_{pp,n} \).
The correlation in each case is stronger for younger ages. This occurs because, whereas $L_{oa}$ and $L_{pp}$ are affected by uncertainty in all future death probabilities, $L_{oa,n}$ and $L_{pp,n}$ are only affected by death probabilities at time $t = 65 - x$.

The above results suggest that exchange rate uncertainty could substantially affect longevity risk. In addition, the effect is likely to depend strongly on the type of pension plan, whether the insured has a partner, and, if $p = 1$, on the couple’s choice between a single life annuity and a joint life annuity in case they are both alive at the time the participant retires (i.e. the parameter $\alpha$). Figure 5 displays $\sigma(L)/E[L]$ as a function of the age of the participant, for the two pension plans, and for three types of insured: an insured who does not have a partner ($p = 0$); a couple that will prefer a joint and survivor annuity in case the partner is still alive ($p = 1, \alpha = 1$), and a couple that will prefer a single life annuity ($p = 1, \alpha = 0$). The left panel is for males and the right panel for females. We now discuss the effect of partnership, preference between single life and joint and survivor annuities, and the type of pension plan, on longevity risk.

The effect of partnership ($p$) and preferences ($\alpha$)
For every age $x$, for both genders, and for both types of pension plans, longevity risk risk is highest for a participant without a partner ($p = 0$). Within coupled participants, risk is lower for those who prefer a joint and survivor annuity ($p = 1, \alpha = 1$).
Figure 5: $\sigma(L)/E[L]$ for $p = 0$ (solid lines), $p = 1$ and $\alpha = 0$ (dashed lines), and $p = 1$ and $\alpha = 1$ (dotted lines), for a participant in a JointLife plan (upper graphs) and for a participant in a SingleLife plan (lower graphs). Left panel: males; right panel: females. In each case, $w = 2/3$.

These results are driven by the fact that, as discussed in Subsection 4.1, partner pension at the age of 65 has a large hedge potential for longevity risk in old-age pension liabilities (see (28)). Note that the differences between the three types of insureds are larger for men than for women. This occurs because the hedge potential of (temporary) partner pension rights for males is larger than for females, due to the higher expected present value of these rights.

The effect of the type of pension plan
For every age $x$, for both genders, and for each of the three types of individuals, longevity risk is higher for the SingleLife plan than for the JointLife plan. This result is driven by the effect of exchange rate uncertainty on the correlation between the level of payments ($\delta_1$ and $\delta_2$), and the duration of payments ($L_{oa}$ and $L_{pp}$). The following table summarizes these correlations. The results follow immediately from Proposition 2, and the fact that
\[ \rho(e, L_{oa}) < 0, \text{ and } \rho(e, L_{pp}) > 0. \]

<table>
<thead>
<tr>
<th>JointLife</th>
<th>SingleLife</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\rho(\delta_1, L_{oa}))</td>
<td>(\rho(\delta_1, L_{oa}))</td>
</tr>
<tr>
<td>(\rho(\delta_2, L_{pp}))</td>
<td>(\rho(\delta_2, L_{pp}))</td>
</tr>
</tbody>
</table>

In case of a *JointLife* plan, the correlation between \(\delta_1\) and \(L_{oa}\) is negative in all three cases, which implies that a higher than expected increase in the duration of old-age payments (due to higher than expected reduction in mortality rates) is partly mitigated by a higher than expected reduction in the level of the nominal old-age pension payment, \(\delta_1\). This hedge effect is not present in case of a *SingleLife* plan. There, levels and durations are uncorrelated, except for the coupled participant who prefers a joint and survivor annuity. For this participant, the effect of higher than expected duration of old-age pension payments is combined with higher than expected nominal payments. While higher than expected duration of partner pension payments is partly mitigated by lower than expected nominal payments, the former effect is dominant.

### 4.3 Effect of uncertainty in the term structure of interest rates

The previous subsection shows that longevity risk affects both the insured and the insurer due to uncertainty in the exchange rate. These results are based on the assumption that the term structure of interest rates is flat at 4% and constant over time. However, the value of the exchange rate, as given in equation (10), is actuarially neutral at the time of exchange and hence also depends on the actual term structure of interest rates at the time of exchange. In this subsection we investigate the importance of interest rate risk, by comparing the setting without interest rate risk with a setting with interest rate risk. In order to incorporate the interest rate risk we use the Vasicek model (Vasicek, 1977). In this model the instantaneous spot interest rate \(r_t\) follows an Ornstein-Uhlenbeck process,

\[
\begin{align*}
    dr_t &= (\theta - \alpha \cdot r_t)dt + \sigma dW(t),
\end{align*}
\]

where \(\alpha, \theta,\) and \(\sigma\) are given constants and where \(W(t)\) represents a standard Wiener process. The model yields as time \(t\) price of a zero coupon bond with time to maturity
\[ P_t^{(\tau)} = \exp(-A_{\tau} - B_{\tau}r_t), \]  

where \( B_{\tau} \) and \( A_{\tau} \) are given by:

\[
B_{\tau} = \frac{1}{\alpha} (1 - \exp(-\alpha \tau)),
\]

\[
A_{\tau} = (\tau - B_{\tau}) \frac{\theta - \lambda \sigma}{\alpha} - \frac{\sigma^2}{4\alpha^3} (4 \exp(-\alpha \tau) - \exp(-2\alpha \tau) + 2\alpha \tau - 3),
\]

with \( \lambda \) representing the interest rate specific market price of risk.

Table 1 displays the value of the parameters of the Vasicek model that we use. In contrast to the modeling of longevity risk where we use process, parameter, and model risk, we only use process risk for modeling interest rate risk. The parameters are such that the current \((r_0)\) as well as the long term average \((\theta/\alpha)\) short rate is set at 4\%, the current 30 years interest rate is set at 5\%, \(\sigma\) is set such that it satisfies the solvency rules for in the Netherlands\(^\text{11}\), and \(\theta\) is set such that in a corresponding AR(1) model of the short term interest rate estimated using monthly data the AR(1) coefficient is 0.96.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>(\lambda)</th>
<th>(\sigma)</th>
<th>(\theta)</th>
<th>(\alpha)</th>
<th>(r_0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>-0.555</td>
<td>0.008</td>
<td>0.016</td>
<td>0.4</td>
<td>0.04</td>
</tr>
</tbody>
</table>

Table 1: Parameters of the Vasicek model.

To quantify the impact of the uncertainty in the term structure, we calculate the quantiles of the distribution of the exchange rate \(e\) and the correlation between the exchange rate and the duration of the payments, comparing the case \(r_t = 0.04\) for all \(t\) (corresponding to \(\sigma = 0\)), the the setting where uncertainty in the short term interest rate is as given by the Vasicek model, using the parameters presented in Table 1.

Figure 6 displays selected quantiles of the distribution of \(e\) as a function of \(x\) and the correlation between \(e\) and \(L_{oa}\) and between \(e\) and \(L_{pp}\) for both genders, also as a function of \(x\). The left panels are without uncertainty in the term structure, the right panels with uncertainty.

\(^{11}\)The solvency rules for a pension fund in the Netherlands (FTK) prescribe that for calculating the buffer a pension fund should incorporate that the one-year interest rates could drop with a factor 0.35. When the interest rate is 0.04, the drop in interest rate is 0.014. Since the solvency rules for pension funds are based on the 97.5\% quantile, this would imply that \(\sigma\) is approximately 0.007.
Figure 6: Upper panels: selected quantiles of the distribution of $e(65 - x)$ as a function of $x$: the median (bold line), the 25% and 75% quantile (dashed lines), and the 10% and 90% quantile (dotted lines). Lower panels: $\rho(e, L_{oa})$ as a function of $x$ (dashed lines) for males (lower graph) and for females (upper graph), and $\rho(e, L_{pp})$ as a function of $x$ (dotted lines) for males (upper graph) and for females (lower graph). Left panels: No uncertainty in the term structure, i.e., $r_t = 0.04$ for all $t$. Right panels: with uncertainty in the term structure, i.e., the process of $r_t$ is given by equation (29), using the parameters of Table 1.

We observe that the uncertainty in the interest rate has a negligible impact on the distribution of $e$ and on the correlation between the exchange rate and the expected present values of the liabilities. The results are obtained using only process uncertainty for the interest rates but including process, parameter, and model risk for longevity uncertainty. However, longevity risk in the exchange rate yields $\sigma(e(65 - x))/E[e(65 - x)] = 1.4\%$ for a 64 years old and the value 8.6% for a 25 years old insured without interest rate risk, while interest rate risk in absence of longevity risk\(^{12}\) yields $\sigma(e(65 - x))/E[e(65 - x)] < 0.4\%$ for all ages between 25 and 64. This suggests that longevity risk will dominate, also when the uncertainty in the interest rates is larger, for instance, due to parameter or model risk.

Figure 7 displays $\sigma(L)/E[L]$ as a function of the age of the participant, for the two pension plans, and for three types of insureds: an insured who does not have a partner ($p = 0$), a couple that will prefer a joint and survivor annuity in case the partner is still alive ($p = 1, \alpha = 1$), and a couple that will prefer a single life annuity ($p = 0$).

\(^{12}\)We took the median mortality probabilities for the calculations.
1, α = 0), where we also consider both genders. The left panels are without interest rate uncertainty and the right panels with uncertainty. Again, we observe that interest rate risk affects the risk in the present values of the pension liabilities only marginally. This is not surprising, given that the interest rate risk affects the exchange rate only marginally.

Figure 7: σ(L)/E[L] for p = 0 (solid lines), p = 1 and α = 0 (dashed lines), and p = 1 and α = 1 (dotted lines), for a participant in a JointLife plan (upper graphs) and for a participant in a SingleLife plan (lower graphs). Upper panels: males; lower panels: females. In each case, w = 2/3. Left panels: No uncertainty in the term structure of a zero coupon bond price, i.e. rt = 0.04 for all t. Right panels: with uncertainty in the term structure of a zero coupon bond price, i.e. the process of rt is given by equation (29).

4.4 Exchange rates and actuarial neutrality

The previous subsection shows that exchange rate uncertainty significantly affects both the insured and the insurer. These results are based on a gender- and age-neutral exchange rate, as defined in (10). Exchange rates that are either age-specific or gender-specific, however, are also not uncommon. Therefore, we have replicated the results of Subsection 4.2 for the case of age-specific and gender-neutral exchange rates, as well as age-neutral and gender-specific exchange rates. The results are reported in Appendix B. While, for example, under gender-specific, but age-neutral exchange rates, the difference in longevity risk between the JointLife and the SingleLife plan is higher for males, and
lower for females, the qualitative results reported in Subsection 4.2 still hold in each case.

However, with gender- and/or age-neutral exchange rates, the expected present value of the pension rights before exchange will typically differ from the expected value of the pension rights after exchange, i.e. the exchange is not actuarially neutral at individual level. Specifically, with gender neutral exchange rates, the value of partner pension rights for a man (woman) is typically underestimated (overestimated). No gender "discrimination" in exchange rates will therefore lead to "discrimination" in the value of pension rights. Therefore, in the remainder of this subsection we investigate the effect on longevity risk of individual-specific exchanges rates. In this case, the exchange rate at date $t$ equals the ratio of the *individual-specific* expected present value of partner pension payments for the individual’s partner, over the expected present value of the old-age pension payments, both conditional on realized survival rates at time $t$ ($P_t$), i.e.

$$e = e(y, g, g', t) = \frac{\mathbb{E}[L_{pp}(65, y + t, g, g', t)|P_t]}{\mathbb{E}[L_{oa}(65, g, t)|P_t]}$$

(31)

so that the exchange rate then is a function of the gender of the insured, the age and gender of the partner, and time. The left panel of Figure 8 displays selected quantiles of $e(y, g, g', 65 - x)$ as a function of the age $x$ of the insured, for males with a female partner aged $y = x - 3$ (lower graph), and for females with a male partner aged $y = x + 3$ (lower graph). The right panel of Figure 8 displays the correlation between the individual-specific exchange rate $e(y, g, g', 65 - x)$ and $L_{oa}$ (dashed lines) for males (lower graph) and for females (lower graph), as well as the correlation between $e$ and $L_{pp}$ (dotted lines) for males (upper graph) and for females (lower graph).

Compared to age- and/or gender-neutral exchange rates, as defined in (10), see Figures 3 and 4, individual-specific exchanges rates:

i) are significantly higher for males, and are significantly lower for females;

ii) are significantly more volatile;

iii) are significantly more positively correlated with $L_{pp}$.

The difference in the level of the exchange rates (i)) is due to higher mortality probabilities for males than for females, leading to a lower (higher) expected present value of old-age pension liabilities, and a higher (lower) expected value of partner pension liabilities for men (women). Gender-neutral exchange rates are less volatile than
Figure 8: Left panel: Selected quantiles of the distribution of \( e = e(y, g, g', 65 - x) \) as a function of \( x \), with \( y = x - 3 \) for males and \( y = x + 3 \) for females: the median (bold line), the 25% and 75% quantile (dashed lines), and the 10% and 90% quantile (dotted lines), for males (upper graphs) and for females (lower graphs). Right panel: \( \rho(e, L_{oa}) \) as a function of \( x \) (dashed lines) for males (lower graph) and for females (upper graph), and \( \rho(e, L_{pp}) \) as a function of \( x \) (dotted lines) for males (upper graph) and for females (lower graph).
individual-specific exchange rates (ii)), because, due to the fact that male and female probabilities are not perfectly correlated, averaging over the two genders creates a hedge effect. For the same reason, the correlation between the gender-neutral exchange rate, and the expected present value of the individual’s old-age pension liabilities/partner pension liabilities ($L_{oa}$ and $L_{pp}$), is weaker than the correlation between individual-specific exchange rates, and $L_{oa}$ and $L_{pp}$. The effect is particularly significant for $L_{pp}$.

These results suggest that the effect of exchange rate uncertainty on longevity risk might be significantly different under individual-specific exchange rates. Figure 9 displays $\sigma(L)/E[L]$ as a function of the age of the participant, for the two pension plans, and for three types of insureds: an insured who does not have a partner ($p = 0$); a couple that prefers a joint and survivor annuity in case the partner is still alive ($p = 1, \alpha = 1$), and a couple that prefers a single life annuity ($p = 1, \alpha = 0$). The left panel is for males with a three years younger female partner, and the right panel for females with a three years older male partner.

We see that the differences between the two plans are larger than under gender-neutral exchange rates. We have shown in the previous subsection that the effect of exchange rate uncertainty on longevity risk, and in particular on the difference between the two
plans, is driven by that fact that exchange rate uncertainty induces correlation between the nominal payments ($\delta_1$ and $\delta_2$) and the duration of the participant’s old-age pension payments/partner pension payments, ($L_{oa}$ and $L_{pp}$), respectively. The correlation between the gender-neutral exchange rate and the (individual-specific) duration of payments is weaker than under individual-specific exchange rates. This implies that the effect of exchange rate uncertainty on the difference between the two plans is larger with individual-specific exchange rates than with gender-neutral exchange rates.

5 Conclusions and further research

This paper investigates the effect of longevity risk on pension plans that, at retirement age, allow the participant to choose between a single life annuity for the participant’s life, or a joint and survivor annuity. We show that these plans are affected by longevity risk in two ways. First, the participant’s choices at retirement age affect product and gender mix. We show that product and gender mix can significantly affect longevity risk. Second, longevity risk is affected due to the fact that, for insureds who are not yet retired, the exchange rate is a random variable that depends on future realizations of death rates. We show that exchange rate uncertainty significantly affects both the participant and the pension fund. For the participant, depending on his preferences and the type of pension plan, it induces uncertainty in the level of the nominal old-age pension rights and/or partner pension rights. For the pension fund, it affects the expected present value of future payments in two ways. First, because nominal insured rights are uncertain, and second because exchange rate uncertainty induces correlation between the level of the payments and the duration of the payments.

Our results suggest that a pension plan where participants accrue both old-age pension rights and partner pension rights, and are allowed to exchange their partner pension rights for additional old-age pension rights, is significantly less affected by longevity risk than the alternative plan where participants accrue only old-age pension rights, and can exchange part of those rights for partner pension rights. However, we also show that the effect of longevity risk for coupled participants depends significantly on their preferences at retirement age, in case they are both alive, between a single life annuity and a joint and survivor annuity. These preferences are likely to be affected by longevity risk. This effect is not investigated in this paper and is left for future research.
References


A Proofs

Proof of Proposition 1. Follows immediately from (20), (14), and (16), and the fact that for \( x \geq 65 \), \( e(65 - x) \) and \( 1_{OA} \) are deterministic.

Proof of Proposition 2. In order to reduce notational complexity, we let \( T_x^{(g)} := T_{x,0}^{(g)} \) and \( T_y^{(g)} := T_{y,0}^{(g)} \).

i) Consider an insured in a JointLife plan, aged \( x < 65 \) with a partner. First consider \( \tau \geq 65 - x \). Then, given (18), and since, conditional on \( \mathcal{P}_\infty \), the random variables \( T_x^{(g)} \), \( T_y^{(g)} \), \( F \), and \( e \) are independent, and since \( \mathbb{E}[e|\mathcal{P}_\infty] = e \), and \( \mathbb{E}[F|\mathcal{P}_\infty] = \alpha \), it follows that:

\[
\mathbb{E} \left[ (1 + w \cdot e \cdot 1_{OA}) \cdot \tilde{L}_{oa}(\tau)|\mathcal{P}_\infty \right] \\
= \mathbb{E} \left[ \left( 1 + e \cdot \left( 1 - F \cdot 1_{(T_y^{(g')} \geq 65-x)} \right) \right) \cdot 1_{(T_x^{(g')} \geq \tau)}|\mathcal{P}_\infty \right] \\
= \left( 1 + e \cdot \left( 1 - \mathbb{E} \left[ F \cdot 1_{(T_y^{(g')} \geq 65-x)}|\mathcal{P}_\infty \right] \right) \right) \cdot \mathbb{E}[1_{(T_x^{(g')} \geq \tau)}|\mathcal{P}_\infty] \\
= \left( 1 + e \cdot \left( 1 - \alpha \cdot \gamma_{65-x} p_{y,0}^{(g')} \right) \right) \cdot \mathbb{E}[\tilde{L}_{oa}(\tau)|\mathcal{P}_\infty]
\]

Similarly,

\[
\mathbb{E} \left[ (1 - 1_{OA}) \cdot \tilde{L}_{pp}(\tau)|\mathcal{P}_\infty \right] \\
= \mathbb{E} \left[ \left( 1_{(T_y^{(g')} \geq 65-x)} \cdot F \right) \cdot \tilde{L}_{pp}(\tau)|\mathcal{P}_\infty \right] \\
= \mathbb{E} \left[ 1_{(65-x \leq T_x^{(g')} < \tau)} 1_{(T_y^{(g')} \geq \tau)} \cdot 1_{(T_y^{(g')} \geq 65-x)} \cdot F|\mathcal{P}_\infty \right] \\
= \mathbb{E} \left[ 1_{(65-x \leq T_x^{(g')} < \tau)} 1_{(T_y^{(g')} \geq \tau)}|\mathcal{P}_\infty \right] \cdot \mathbb{E}[F|\mathcal{P}_\infty] \\
= \alpha \cdot \mathbb{E}[\tilde{L}_{pp}(\tau)|\mathcal{P}_\infty].
\]

Thus, for \( \tau \geq 65 - x \), (14) implies

\[
\mathbb{E}[\tilde{L}(\tau)|\mathcal{P}_\infty] = \left( 1 + e \left( 1 - \alpha \cdot \gamma_{65-x} p_{y,0}^{(g')} \right) \right) \cdot \mathbb{E}[\tilde{L}_{oa}(\tau)|\mathcal{P}_\infty] \\
+ w \cdot \alpha \cdot \mathbb{E}[\tilde{L}_{pp}(\tau)|\mathcal{P}_\infty] + w \cdot \mathbb{E}[\tilde{L}_{pp}(\tau)|\mathcal{P}_\infty].
\]

For \( \tau < 65 - x \), it holds that \( \tilde{L}_{oa}(\tau) = \tilde{L}_{pp}(\tau) = 0 \), and so \( \tilde{L}(\tau) \), the liability payment in period \( \tau \) consists of a temporary partner payment, and since liabilities for temporary partner pensions are not affected by exchange behavior or exchange rates, it holds that:

\[
\mathbb{E}[\tilde{L}(\tau)|\mathcal{P}_\infty] = w \cdot \mathbb{E}[\tilde{L}_{pp}(\tau)|\mathcal{P}_\infty]
\]
Therefore, it follows from (20), (33), and (34) that
\[
\sum_{\tau=0}^{110-x} \mathbb{E} \left[ \tilde{L}(\tau) | \mathcal{P}_\infty \right] P(\tau) = \left( 1 + e \left( 1 - \alpha \cdot 65 - x \cdot L(g') \right) \right) \sum_{\tau=0}^{110-x} \mathbb{E} \left[ \tilde{L}_{oa}(\tau) | \mathcal{P}_\infty \right] P(\tau) \\
+ w \cdot \alpha \sum_{\tau=65-x}^{110-x} \mathbb{E} \left[ \tilde{L}_{pp}(\tau) | \mathcal{P}_\infty \right] P(\tau) + w \sum_{\tau=0}^{110-x} \mathbb{E} \left[ \tilde{L}_{pp}(\tau) | \mathcal{P} \right] P(\tau),
\]
so that (21) is indeed satisfied.

The proof for the SingleLife plan is similar.

ii) Follows immediately from (14), (16), (20), and the fact that \( \mathbb{E} [e | \mathcal{P}_\infty] = e \).

**Proof of Proposition 3.** Consider the JointLife plan. Then, the aggregate liabilities of the fund are given by:
\[
L = \sum_{i \in M} C_i \cdot \left[ (1 + w \cdot e \cdot 1_{OA}(i)) \cdot L_{oa,m} + w \cdot (1 - 1_{OA}(i)) \cdot L_{pp,m} \right] \\
+ \sum_{i \in F} C_i \cdot \left[ (1 + w \cdot e \cdot 1_{OA}(i)) \cdot L_{oa,m} + w \cdot (1 - 1_{OA}(i)) \cdot L_{pp,m} \right].
\]
Equivalently,
\[
L = \left( \sum_i C_i \right) \cdot \left[ \delta_{1,m} \cdot L_{oa,m} + \delta_{1,f} \cdot L_{oa,f} + \delta_{2,m} \cdot L_{pp,m} + \delta_{2,f} \cdot L_{pp,f} \right],
\]
where
\[
\begin{align*}
\delta_{1,m} &= \gamma \cdot (1 + (1 - \tilde{\alpha}_m) \cdot w \cdot e); \\
\delta_{1,f} &= (1 - \gamma) \cdot (1 + (1 - \alpha_f) \cdot w \cdot e); \\
\delta_{2,m} &= \gamma \cdot \tilde{\alpha}_m \cdot w; \\
\delta_{2,f} &= (1 - \gamma) \cdot \tilde{\alpha}_f \cdot w.
\end{align*}
\]
Note that
\[
L = (\delta_{1,m} + \delta_{1,f}) \cdot \left( \sum_i C_i \right) \cdot \left[ \frac{\delta_{1,m}}{\delta_{1,m} + \delta_{1,f}} \left( L_{oa,m} + \frac{\delta_{2,m}}{\delta_{1,m}} \cdot L_{pp,m} \right) + \left( 1 - \frac{\delta_{1,m}}{\delta_{1,m} + \delta_{1,f}} \right) \cdot \left( L_{oa,f} + \frac{\delta_{2,f}}{\delta_{1,f}} \cdot L_{pp,f} \right) \right]
\]
Since \( (\delta_{1,m} + \delta_{1,f}) \cdot (\sum_i C_i) \) is deterministic, this concludes the proof. The proof for the case of a SingleLife plan is similar. ■
B  **Effect of exchange rate definition**

In this Appendix, we replicate the results of Subsection 4.2 for alternative definitions of the exchange rate. Let

\[
L_{oa,e}(g, t) := \mathbb{E}[L_{oa}(65, g, t) | \mathcal{P}_t],
\]

\[
L_{pp,e}(g, g', d, t) := \mathbb{E}[L_{pp}(65, 65 + d, g, g', t) | \mathcal{P}_t],
\]

denote the date-\(t\) expected present value, conditional on \(\mathcal{P}_t\), of old-age pension payments for a 65-year old participant with gender \(g\), and the expected present value, conditional on \(\mathcal{P}_t\), of partner pension payments for a 65-year old participant with gender \(g\) with a partner with gender \(g'\), who is \(d\) years older. Gender \(g = n\) refers to the use of gender-neutral probabilities as defined in (9). Then, we consider the following four cases:

i) *gender-neutral, age-specific* exchange rates based on gender-neutral, age-specific liability values:

\[
e_{n,s}(d, t) := \frac{L_{pp,e}(n, n, d, t)}{L_{oa,e}(n, t)};
\]

ii) *gender-neutral, age-specific* exchange rates based on gender- and age-specific liability values:

\[
\hat{e}_{n,s}(d, t) = \frac{\frac{1}{2}L_{pp,e}(m, f, d, t) + \frac{1}{2}L_{pp,e}(f, m, d, t)}{\frac{1}{2}L_{oa,e}(m, t) + \frac{1}{2}L_{oa,e}(f, t)};
\]

iii) *gender- and age-neutral* exchange rates based on gender- and age-neutral liability values (the case treated in Subsection 4.2):

\[
e_{n,n}(t) := \frac{e_{n,s}(-3, t) + e_{n,s}(3, t)}{2};
\]

iv) *gender- and age-neutral* exchange rates based on gender-specific and age-neutral liability values:

\[
\hat{e}_{n,n}(t) = \frac{\hat{e}_{n,s}(-3, t) + \hat{e}_{n,s}(3, t)}{2}.
\]
Figure 10: $\sigma(L)/E[L]$ for $p = 0$ (solid lines), $p = 1$ and $\alpha = 0$ (dashed lines), and $p = 1$ and $\alpha = 1$ (dotted lines), for a participant in a JointLife plan (upper graph) and for a participant in a SingleLife plan (lower graph). Left panels: males; right panels: females. Upper panels: $e_{n,s}(d,t)$; Middle panels: $\hat{e}_{n,s}(d,t)$; Lower panels: $\hat{e}_{n,n}(t)$. In each case, $w = 2/3$. 
C Forecasting future mortality

In this appendix we briefly describe the models used to quantify the macro-longevity risk affecting \( p_{x,t}^{(g)} \). Let \( \mu_{x,t}^{(g)} \) denote the force of mortality of a person with age \( x \) and gender \( g \) at time \( t \), i.e.,

\[
\mu_{x,t}^{(g)} = \lim_{\Delta t \to 0} \frac{P \left( 0 \leq T_{x,t}^{(g)} \leq \Delta t \right)}{\Delta t}.
\]

We assume that for any integer age \( x \), any gender \( g \), and any time \( t \), it holds that \( \mu_{x+u,t}^{(g)} = \mu_{x,t}^{(g)} \), for all \( x \in [0,1) \). Then, one can verify

\[
p_{x,t}^{(g)} = \exp \left( -\mu_{x,t}^{(g)} \right).
\]

Next, let \( D_{x,t}^{(g)} \) denote the observed number of deaths in year \( t \) in a cohort with gender \( g \) and aged \( x \) at the beginning of year \( t \), and let \( E_{x,t}^{(g)} \) denote the number of person years during year \( t \) in a cohort with gender \( g \) and aged \( x \) at the beginning of year \( t \), the so-called exposure. Then, under appropriate regularity conditions, as discussed by Gerber (1997), the Maximum Likelihood estimator for the force of mortality is given by

\[
\hat{\mu}_{x,t}^{(g)} = \frac{D_{x,t}^{(g)}}{E_{x,t}^{(g)}}.
\]

We assume that there is no (sampling) risk involved in this relationship, so that the macro-longevity risk is fully captured by the risk in \( D_{x,t}^{(g)}/E_{x,t}^{(g)} \). We use the models proposed by Lee and Carter (1992), Brouhns, Denuit, and Vermunt (2002), and Cossette et al. (2007) to quantify the macro-longevity risk in \( D_{x,t}^{(g)}/E_{x,t}^{(g)} \). The model by Lee and Carter (1992) is given by

\[
\log \left( \frac{D_{x,t}^{(g)}}{E_{x,t}^{(g)}} \right) = a_x^{(g)} + b_x^{(g)} k_t^{(g)} + \epsilon_{x,t}^{(g)},
\]

where \( k_t^{(g)} \) is an index of the level of mortality, \( a_x^{(g)} \) is an age-specific constant describing the general pattern of mortality by age, \( b_x^{(g)} \) is an age-specific constant describing the relative speed of the change in mortality by age, and where \( \epsilon_{x,t}^{(g)} \) represents the measurement error, assumed to satisfy \( \epsilon_{x,t}^{(g)} | K_t \sim N(0, \sigma_{x,g}^2) \), conditional on \( K_t = \{ k_{\tau}^{(g)} | g \in \{m,f\}, \tau = t, t-1, \ldots \} \). Moreover, we assume that the \( \epsilon_{x,t}^{(g)} \) are independent for different \( x \) and \( g \), conditional on \( K_t \).

To model the process for \( \left(k_t^{(m)}, k_t^{(f)}\right) \) over time, we use an ARIMA(0,1,1) model

\[
k_t^{(m)} = k_{t-1}^{(m)} + c_t^{m} + \theta_t^{m} \epsilon_{t-1}^{(m)} + \epsilon_t^{m},
\]

\[
k_t^{(f)} = k_{t-1}^{(f)} + c_t^{f} + \theta_t^{f} \epsilon_{t-1}^{(f)} + \rho \epsilon_t^{m} + \epsilon_t^{f},
\]

where \( c_t^{g} \) is the gender \( g \) specific drift term which indicates the average annual change of \( k_t^{(g)} \), \( \theta_t^{g} \) is the gender specific moving average coefficient, and \( \epsilon_t^{g} \) is the gender specific
The reduction factor is given by 

\[
\begin{pmatrix}
1 \\
\hat{c}_t
\end{pmatrix} \big| k_{t-1} \sim N \left( \begin{pmatrix}
0 \\
0
\end{pmatrix}, \begin{pmatrix}
\sigma_m^2 & 0 \\
0 & \sigma_f^2
\end{pmatrix} \right).
\]

The parameter \( \rho \) captures the correlation between \( k_{t}^m \) and \( k_{t}^f \) over time.

In case of the model by Brouhns, Denuit, and Vermunt (2002), the age and gender specific numbers of deaths are modeled by a Poisson process,

\[
D_{x,t}^{(g)} \sim \text{Poisson} \left( E_{x,t}^{(g)} e^{a_{x,T}^{(g)} + b_{x,T}^{(g)} k_{t}^{(g)}} \right),
\]

with \( \tilde{K}_t = K_t \bigcup \{ E_{x,t}^{(g)} \mid g \in \{ m, f \}, \text{all } x, t = t - 1, \ldots \} \). We assume that the \( D_{x,t}^{(g)} \) are independent for different \( x \) and \( g \), conditional on \( \tilde{K}_t \). The process for \( (k_{t}^m, k_{t}^f) \) is modeled as in case of the Lee and Carter (1992)-model, i.e., via equations (37)–(38).

As third model, we consider Cossette et al. (2007). These authors model the age specific numbers of deaths \( D_{x,t}^{(g)} \) via the Binomial Gumbel process,

\[
D_{x,t}^{(g)} \mid \tilde{K}_t \sim \text{Bin} \left( E_{x,t}^{(g)}, 1 - \exp \left( -e^{a_{x,T}^{(g)} + b_{x,T}^{(g)} k_{t}^{(g)}} \right) \right),
\]

where we again assume that the \( D_{x,t}^{(g)} \) are independent for different \( x \) and \( g \), conditional on \( \tilde{K}_t \), and where we model the process for \( (k_{t}^m, k_{t}^f) \) via equations (37)–(38).

To forecast the future mortality rates, we use the current time (defined in the appendix as time \( T \)) mortality table and a reduction factor. In this way the mortality rates are set such that the estimation error in the last year of the mortality data is zero, so that we avoid a jump-off bias in the forecasts. Let \( q_{x,T}^{(g)} = 1 - p_{x,T}^{(g)} \), be based on the last year of mortality data. Then we forecast \( q_{x,T+s}^{(g)} \) as follows

\[
q_{x,T+s}^{(g)} = q_{x,T}^{(g)} \times \overline{RF}_{x,T,s}^{(g)}.
\]

The reduction factor is given by

\[
\overline{RF}_{x,T,s}^{(g)} = e^{\hat{b}_x^{(g)} \times (\hat{k}_{T+s}^{(g)} - \hat{b}_x^{(g)})},
\]

where \( \hat{b}_x^{(g)} \) and \( \hat{k}_{T+s}^{(g)} \) denote the (model specific) estimated \( b_{x,T}^{(g)} \) and \( k_{T}^{(g)} \), respectively, and where \( \hat{k}_{T+s}^{(g)} \) denotes the \( s \geq 1 \) periods ahead forecast. For the latter we use

\[
\begin{pmatrix}
\hat{k}_{T+s}^m \\
\hat{k}_{T+s}^f \\
\hat{k}_{T+s-1}^m \\
\hat{k}_{T+s-1}^f
\end{pmatrix} \big| \tilde{K}_{T+s-1} \sim N \left( \begin{pmatrix}
\hat{k}_{T+s-1}^m + \hat{c}_m \\
\hat{k}_{T+s-1}^f + \hat{c}_m
\end{pmatrix}, \begin{pmatrix}
\hat{\sigma}_m^2 & 0 \\
0 & \hat{\sigma}_m^2
\end{pmatrix} \right)
\]

\right).
with $\hat{K}_T = K_T$, and $\hat{K}_{T+s} = \hat{K}_{T+s-1} \cup \{\hat{k}^g_{T+s}, \hat{k}^f_{T+s}\}$, for $s = 1, 2, 3, \ldots$, employing (model specific) estimates.

In order to avoid localized age induced anomalies in $\hat{b}_x^{(g)}$ in the three models, we follow Renshaw and Haberman (2003). These authors proposed to smooth the age specific estimated parameters $\hat{b}_x^{(g)}$ using cubic B-splines, with internal knots,

$$
\zeta_0^{(g)} + \zeta_1^{(g)} x + \zeta_2^{(g)} x^2 + \zeta_3^{(g)} x^3 + \sum_{j=1}^r \zeta_3^{(g)} (x - x_j)^3_+,
$$

where $(x - x_j)^3_+ = (x - x_j)^3$, in case $x - x_j > 0$, and zero otherwise. As internal knots we use $x_1 = 9.5$, $x_2 = 20.5$, $x_3 = 50.5$, $x_4 = 60.5$, and $x_r = x_5 = 80.5$. The cubic B-splines are fitted to the (model specific) estimated $\hat{b}_x^{(g)}$ using the method of least squares.

The model-specific parameters are estimated imposing the required normalizations and using the estimation techniques as described in the corresponding papers. For the Lee and Carter (1992)-model we first estimate the parameters $a_x^{(g)}$, $b_x^{(g)}$, and $k_t^{(g)}$ using a singular value decomposition (SVD). Secondly, for all $t \leq T$ and $g \in \{m, f\}$, we reestimate $k_t^{(g)}$ such that the estimated number of deaths using the estimates of $a_x^{(g)}$ and $b_x^{(g)}$ in equation (1) (with $\epsilon_x^{(g)} = 0$) equals the observed number of deaths. These reestimated $k_t^{(g)}$, $t \leq T$, are used to estimate the process for $\left(k_t^{(m)}, k_t^{(f)}\right)$ using equations (37)–(38). For the Brouhns, Denuit, and Vermunt (2002)-model and the Cossette et al. (2007)-model we use the iterative procedure proposed by Goodman (1979) to obtain the Maximum Likelihood estimates, where the criterium to stop the procedure is a very small (i.e., $10^{-10}$) increase of the log-likelihood.

Age, gender, and time specific numbers of death and exposed to death are obtained from the Human Mortality Database. In our case $x \in \{0, 1, 2, \ldots, 99, 100^+\}$, with $100^+$ the age group of people aged 100 years or more. We use the time period 1977–2004, so that $T = 2004$. This time period minimizes the statistic proposed by Booth et al. (2002) to test the hypothesis that the age components ($b_x^{(g)}$) are invariant over time. We use this selection, since mortality experience in the industrialized world seems to suggest a substantial age-time interaction in the twentieth century.

The parameter estimates relevant for the quantification of the macro-longevity risk are plotted in Figure 1 (the $\hat{b}_x^{(g)}$) and Table I (the parameter estimates of equations (37)–(38)). The estimation results for the different models are quite comparable, with the estimation results of the Lee and Carter (1992) model slightly deviating from the other two models. At younger ages males and females are more sensitive to changes in the time trend, while also males around 60 and females around 77 show an increased sensitivity.

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13See www.mortality.org.
Table 2: Estimation results

<table>
<thead>
<tr>
<th>Model</th>
<th>$g$</th>
<th>$c^{(g)}$</th>
<th>$\theta^{(g)}$</th>
<th>$\sigma_g$</th>
<th>$\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lee-Carter</td>
<td>$m$</td>
<td>-1.6923</td>
<td>-0.4196</td>
<td>1.0628</td>
<td>0.4545</td>
</tr>
<tr>
<td></td>
<td>$f$</td>
<td>-1.2902</td>
<td>-0.6056</td>
<td>1.3475</td>
<td></td>
</tr>
<tr>
<td>Brouhns, Denuit, and Vermunt</td>
<td>$m$</td>
<td>-1.60771</td>
<td>-0.3386</td>
<td>0.7318</td>
<td>0.5873</td>
</tr>
<tr>
<td></td>
<td>$f$</td>
<td>-1.2572</td>
<td>-0.5286</td>
<td>0.9491</td>
<td></td>
</tr>
<tr>
<td>Cossette et al.</td>
<td>$m$</td>
<td>-1.6063</td>
<td>-0.3449</td>
<td>0.7270</td>
<td>0.5944</td>
</tr>
<tr>
<td></td>
<td>$f$</td>
<td>-1.2535</td>
<td>-0.5273</td>
<td>0.9321</td>
<td></td>
</tr>
</tbody>
</table>


see Figure 1. In terms of the $k_t^{(g)}$-processes, we find that the male drift term is more negative than the female drift term, but in case of females the first order moving average term is more negative. The risk in the female process (given by $\hat{\sigma}_f^2 + \hat{\sigma}_m^2$) is also substantial higher than in case of the males (given by $\hat{\sigma}_m^2$). Finally, there is a substantial correlation between the male and female process.

We include three sources of macro longevity risk: process risk, parameter risk, and model risk. First, using (41) and (42), given a specific model and given the corresponding model specific estimates, there is process risk due to fact that future values of $\hat{k}_T^{(g)}$ are risky, see equation (43). Next, given a specific model, the forecasts (41)–(43) are based on model specific estimates, and these estimates are sensitive to estimation inaccuracy. The corresponding risk is referred to as parameter risk. Finally, different models might be used to calculate the forecasts (41)–(43). Assuming that some prior distribution is used to do the forecast calculations, there is also model risk.

To quantify the macro-longevity risk, we proceed as follows. Given the initial data

$$\{ (D_{x,t}^{(g)}, E_{x,t}^{(g)}) \mid x \in \{0, 1, 2, ..., 99, 100^+\}, g \in \{m, f\}, t \in \{1977, ..., 2004\} \},$$

the following steps are taken.

1) For each of the three models, the parameters $\hat{a}_x^{(g)}, \hat{b}_x^{(g)},$ and $\hat{k}_T^{(g)}$ are estimated and the corresponding residuals $r_x^{(g)}$ are computed. Let $R_t$ be the matrix with components $R_{x,t}^g$, for $g \in \{m, f\}, x \in \{0, ..., 100^+\}$.

2) Next, for each model, we generate $B = 5000$ replications $R_t(b)$, $b = 1, ..., B$, of the residual matrix $R_t$, by sampling with replacement. Using these residual matrices,

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14 In case of the Brouhns, Denuit, and Vermunt (2002)- and the Cossette et al. (2007)-model, we calculated the deviance residuals.

Estimated $b_x^{(g)}$ after smoothing using cubic B-splines. Left panel: $g = m$; right panel: $g = f$. The solid curve corresponds to the Lee and Carter (1992)-model; the dashed curve corresponds to the Brouhns, Denuit, and Vermunt (2002)-model, and the dotted curve corresponds to the Cossette et al. (2007)-model.
the corresponding (model specific) bootstrapped numbers of death $\overline{D}^{(g)}_{x,t}(b)$, $b = 1, ..., B$, are determined.\textsuperscript{16}

3) Given the bootstrapped numbers of death $\overline{D}^{(g)}_{x,t}(b)$, we compute the (model specific) bootstrap estimates $\hat{\alpha}^{(g)}_{x}(b)$, $\hat{\beta}^{(g)}_{x}(b)$, $\hat{\kappa}^{(g)}_{t}(b)$, $b = 1, ..., B$, using the described estimation techniques.

4) Given the bootstrap estimates $\hat{\alpha}^{(g)}_{x}(b)$, $\hat{\beta}^{(g)}_{x}(b)$, $\hat{\kappa}^{(g)}_{t}(b)$, we generate $\hat{\kappa}^{m}_{T+s}(b)$ and $\hat{\kappa}^{f}_{T+s}(b)$, using the model specific version of (43), for $s = 1, ..., 85$ and $b = 1, ..., B$. This allows us to calculate the corresponding $P^{(g)}_{x,T+s}(b)$ via (41) and (42), resulting in $P_{t}(b)$ for appropriate $t$.

5) Finally, for some quantity of interest $F = F(P_{t})$, we calculate the (model specific) bootstrap values $F(b) = F(P_{t}(b))$, for $b = 1, ..., B$, for each of the three models. On the basis of the distribution of all bootstrap values of $F(b)$, merged over the three models, we are able to quantify the macro-longevity risk.

\textsuperscript{16}In case of deviance residuals, this requires the use of the inverse relationship between numbers of death and deviance residuals.