Bubble Analysis in Noncausal Processes

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Objective: Predict a bubble outset and/or bubble burst
Methodology:
- Univariate noncausal autoregressive process
- The closed-form functional estimator of predictive density
- A pattern recognition method for detecting the outset and predicting the burst of a bubble based on a sequence of 2-dimensional indicators
Univariate Noncausal Process Forecasting from MAR(1,1) 
Application: Bubble Analysis 

Bitcoin/USD exchange rates 2017-2019

Bitcoin/USD Rate


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Asymmetric bubbles
Outline

1. Univariate Noncausal Process
2. Forecasting from MAR(1,1)
3. Application: Bubble Analysis
Univariate Noncausal Process

1. Forecasting from MAR(1,1)

2. Application: Bubble Analysis

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Bubble Analysis in Noncausal Processes
Noncausal Processes

Let $y_t = y_t^* - m$, where $m$ is the mean or median.

- stationary Causal Autoregressive Process:

  $$y_t = \rho y_{t-1} + \varepsilon_t,$$

  where $(\varepsilon_t)$ is a Gaussian WN and $|\rho| < 1$.

- stationary Noncausal Autoregressive Process:

  $$y_t = \rho y_{t+1} + \varepsilon_t,$$

  where $(\varepsilon_t)$ is i.i.d., non-Gaussian and $|\rho| < 1$.  

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Bubble Analysis in Noncausal Processes
Simulations: \( Y_t = 0.5 Y_{t-1} + \varepsilon_t, \)  
\((\varepsilon_t) i.i.d. \sim \mathcal{N}(0,1)\)
Simulations: $Y_t = 0.5 Y_{t+1} + \varepsilon_t, \quad (\varepsilon_t) \text{i.i.d.} \sim \mathcal{N}(0, 1)$

bubble (local trend)
Bitcoin/USD exchange rates
Simulations: \( Y_t = 0.5 Y_{t+1} + \varepsilon_t, \quad Y_t = 0.5 Y_{t-1} + \varepsilon_t \)
Let $y_t = y_t^* - m$. The mixed (causal-noncausal) autoregressive process $\text{MAR}(s,r)$ is the stationary solution of:

$$\Psi(L^{-1})\Phi(L)y_t = \varepsilon_t,$$

where $\varepsilon_t$ are iid, $\Psi$ and $\Phi$ are polynomials of degrees $s$ and $r$, with roots outside the unit circle. For $r = s = 1$, we get the $\text{MAR}(1,1)$:

$$(1 - \psi L^{-1})(1 - \phi L)y_t = \varepsilon_t.$$  

The causal and noncausal autoregressive parameters: $|\phi| < 1$ and $|\psi| < 1$. 

**Analysis**

- **Definition**: Mixed autoregressive process $\text{MAR}(s,r)$
- **Causal and noncausal parameters**: $|\phi| < 1$ and $|\psi| < 1$
- **Application**: Bubble Analysis in Noncausal Processes
Example: $Y_t$ is a noncausal Cauchy AR(1), i.e. MAR(0,1) with Cauchy errors

- $Y_t$ has no marginal moments: $E|Y_t| = \infty$, no forward conditional moments $E(|Y_t||Y_{t+1}) = \infty$.
- Its past conditional moments exist $E(|Y_t||Y_{t-1}) < \infty$.
- In particular, $E(Y_t|Y_{t-1}) = Y_{t-1}$ is a martingale. $Y_t$ is a stationary process with a unit root that creates a local trend.
Filtering

• error term:

\[ \varepsilon_t = (1 - \psi L^{-1})(1 - \phi L)y_t \]  \hspace{1cm} (2)

• noncausal component \( u_t \) with respect to \( \varepsilon_t \):

\[ u_t = y_t - \phi y_{t-1}, \quad \text{and} \quad u_t - \psi u_{t+1} = \varepsilon_t \] \hspace{1cm} (3)

• causal component \( v_t \) with respect to \( \varepsilon_t \):

\[ v_t = y_t - \psi y_{t+1} \quad \text{and} \quad v_t - \phi v_{t-1} = \varepsilon_t \] \hspace{1cm} (4)

Lanne, Saikkonen (2011) \( u_t \) determine bubble growth, \( v_t \) determine bubble decline
Univariate Noncausal Process
Forecasting from MAR(1,1)
Application: Bubble Analysis

MAR(1,1) simulation $\phi = 0.3, \psi = 0.9$
Latent components, simulation $\phi = 0.3, \psi = 0.9$
Univariate Noncausal Process

Forecasting from MAR(1,1)

Application: Bubble Analysis
To predict the future value of $y$, we use equivalence conditions of latent variables to predict the future values of component $u$, by finding the conditional p.d.f:

$$l(u_{T+1}, \ldots, u_{T+H} | y_{1}, \ldots, y_{T})$$

$$= l(u_{T+1}, \ldots, u_{T+H} | v_{1}, \epsilon_{2}, \ldots, \epsilon_{T-1}, u_{T})$$

$$= l(u_{T+1}, \ldots, u_{T+H} | u_{T}),$$

(5)

given that $u_{T}$ is independent of the latent components $v_{1}$ and errors $\epsilon_{2}, \ldots, \epsilon_{T-1}$ in the information set.

The predictive density estimator is obtained as follows.
The one-step ahead predictive density of $u_{T+1}$ is:

$$\pi(u_{T+1}|u_T) = \frac{1}{\pi} \frac{1}{1 + (u_T - \psi u_{T+1})^2} \frac{1 + (1 - \psi)^2 u_T^2}{1 + (1 - \psi)^2 u_{T+1}^2},$$

The estimator of the predictive density can be used to forecast $y_{T+1}$ as follows. Given that

$$y_{T+1} = u_{T+1} + \phi y_T,$$

one can compute the predictive density of $y_{T+1}$, $\hat{\pi}(y_{T+1}|y_T, \hat{u}_T)$, by shifting $\hat{\pi}(u_{T+1}|\hat{u}_T)$ by $\phi y_T$.

A similar approach can be applied to forecast the trajectory up to any horizon $h$. 
Predictive densities $\phi = 0.3$, conditioning values

- $y(T)=1.77$, psi=0.0
- $y(T)=25.03$, psi=0.3
- $y(T)=70.18$, psi=0.5
- $y(T)=253.94$, psi=0.9
The **two-step ahead** predictive density is:

\[
\pi(u_{T+1}, u_{T+2} \mid u_T) = \frac{1}{\pi^2} \frac{1}{1 + (u_T - \psi u_{T+1})^2} \frac{1}{1 + (u_{T+1} - \psi u_{T+2})^2} \frac{1}{1 + (1 - |\psi|)^2 u_T^2} \frac{1}{1 + (1 - |\psi|)^2 u_{T+2}^2}.
\]

The two-step ahead predictive density of \( y_{T+1}, y_{T+2} \) is obtained by shifting \( \pi(u_{T+1}, u_{T+2} \mid u_T) \) by \( [\phi y_T, \phi^2 y_T] \).
Predict the joint density of $y_{T+1}$ and $y_{T+2}$ conditional on $y_T = 16.67$, $y_{T-1} = 14.27$, $u_T = 12.39$. 
Joint predictive density for $H=1$ and $H=2$, simulation

$\phi = 0.3, \psi = 0.9$
Joint predictive density for $H=1$ and $H=2$, simulation

$\phi = 0.3$, $\psi = 0.9$
The joint predictive density has a non-Gaussian pattern with long tails (extremes) in some directions.

The semi orthant indicates the direction of change in the future value of $y$.

The distance between the contour lines indicates the rate of change between $y_T$, $y_{T+1}$, and $y_{T+2}$. 
Joint predictive density for $H=1$ and $H=2$, simulation

$\phi = 0.3$, $\psi = 0.9$
Joint predictive density for $H=1$ and $H=2$, simulation

$\phi = 0.3, \psi = 0.9$

<table>
<thead>
<tr>
<th>pattern</th>
<th>probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_T &lt; y_{T-1} &lt; y_{T+1} &lt; y_{T+2}$</td>
<td>0.176</td>
</tr>
<tr>
<td>$y_{T-1} &lt; y_T, y_T &gt; y_{T+1} &gt; y_{T+2}$</td>
<td>0.132</td>
</tr>
<tr>
<td>$y_{T-1} &lt; y_T &lt; y_{T+1}, y_{T+2} &lt; y_{T+1}$</td>
<td>0.520</td>
</tr>
<tr>
<td>$y_{T-1} &lt; y_T &lt; y_{T+1}, y_{T+2} &lt; y_{T+1}, y_{T+2} &lt; y_{T-1}$</td>
<td>0.008</td>
</tr>
</tbody>
</table>
Let us consider the MAR(0,1) process with Cauchy distributed errors:

\[ y_t = \psi y_{t+1} + \epsilon_t, \quad t = 1, \ldots, T. \]

The isodensity curve is enclosed inside ellipse 1 defined by:

\[ (y_t - \psi y_{t+1})^2 + (1 - \psi)^2 y_{t+1}^2 = c - 1, \quad (6) \]

and outside ellipse 2 defined by:

\[ (y_t - \psi y_{t+1})^2 + (1 - \psi)^2 y_{t+1}^2 = 2(\sqrt{c} - 1), \quad (7) \]

where \( c - 1 > 0 \). Since \( c - 1 - 2(\sqrt{c} - 1) = (\sqrt{c} - 1)^2 \geq 0 \), ellipse 1 is outside ellipse 2.
The shape of bivariate predictive density

The isodensity curve is tangent to ellipse 1 at points such that:

\[ y_{t+1} = 0, \text{ or } y_t - \psi y_{t+1} = 0, \]

and is tangent to ellipse 2 at points such that:

\[ y_t = y_{t+1}, \text{ or } y_t = (2\psi - 1)y_{t+1}. \]

When the noncausal autoregressive coefficient \( \psi = 0 \), ellipses 1 and 2 are circles centered at the origin:

\[ y_t^2 + y_{t+1}^2 = c - 1, \text{ and } y_t^2 + y_{t+1}^2 = 2(\sqrt{c} - 1), \]
Isodensity Curve, $\psi = 0$
Isodensity curves for MAR(0,1) process with coefficient $\psi = 0.3$:

The points of tangency of the isodensity are at the intersection of ellipse 1 and lines $y_{t+1} = 0$ and $y_t = 0.3y_{t+1}$ and ellipse 2 and lines $y_t = y_{t+1}$ and $y_t = -0.4y_{t+1}$. 
Isodensity Curve, $\psi \neq 0$
Univariate Noncausal Process
Forecasting from MAR(1,1)
Application: Bubble Analysis
Deformation of bivariate predictive density, 
\[ \phi = 0.3, \psi = 0, y_{T-1} = 2.34, y_T = 1.77 \]
Deformation of bivariate predictive density,
$\phi = 0.3, \psi = 0.3, y_{T-1} = 9.85, y_T = 25.03$
Deformation of bivariate predictive density,
\( \phi = 0.3, \psi = 0.5, y_{T-1} = 37.43, y_T = 70.18 \)
Deformation of bivariate predictive density, 
\( \phi = 0.3, \psi = 0.9, y_{T-1} = 230.89, y_T = 253.94 \)
Let the rate of change at horizon $h$ be denoted by 
$$Z_{T+h} = \frac{y_{T+h}}{y_T}. $$

We study the asymptotic behavior of the rate of change for large $|y_T|$, when the trajectory of the process reaches a high point on a positive bubble.

Assume a fixed $H$ and $\psi > 0$.

The density of $Z_{T+h}$ conditional on $y_T$ is:

$$I_h(Z_{T+h}|y_T) = \frac{y_T}{\pi} \frac{1 - \psi}{1 - \psi^h} \left[ 1 + \frac{(1 - \psi)^2}{(1 - \psi^h)^2} y_T^2 (1 - \psi^h z_{T+h})^2 \right]^{-1}$$

$$= \frac{1 + (1 - \psi)^2 y_T^2}{1 + (1 - \psi)^2 y_T^2 z_{T+h}^2}. $$
The behavior of the predictive density of rate of change $z_{T+H}$ for extremely large values of the process, i.e. when $|y_T| \to \infty$. We distinguish two cases:

i) If $z_{T+h} \neq 0$ and $z_{T+h} \neq 1/\psi^h$, we have

$$l_h(z_{T+h}|y_T) = O(1/y_T) \to 0.$$  

ii) When $z_{T+h} = 0$, or $z_{T+h} = 1/\psi^h$, then

$$l_h(z_{T+h}|y_T) = O(y_T) \to \infty.$$  

Thus, when $|y_T| \to \infty$, the predictive density $l_h$ becomes degenerate. Its mass splits between points 0 (for burst) and $1/\psi^h$ (for growth).
Cauchy Forecasts for large $|y_T|$, H fixed, MAR(0,1)

The probability of burst at point 0 is $p_h(\psi)$ and of growth at point $1/\psi^h$ is $1 - p_h(\psi)$, where

$$p_h(\psi) = \frac{(1 - \psi^h)^2}{\psi^{2h} + (1 - \psi^h)^2}.$$

It is a decreasing function of noncausal autoregressive parameter $\psi$: $p_h(\psi) = 1$ if $\psi = 0$, and $p_h(\psi) = 0$, if $\psi = 1$.

- When noncausal persistence is very high ($\psi \approx 1$), the bubble grows slowly and the probability of bubble burst is low.
- When noncausal persistence is low ($\psi \approx 0$), the bubble grows very fast, and the probability of bubble burst is high.
Application to the Bitcoin/US Dollar Exchange rates

We study a sample of daily values of the Bitcoin-US Dollar exchange rates recorded between December 1, 2018 and May 30, 2019 of length 181, detrended or "demeaned". The Approximate Maximum Likelihood (AML) :

\[ (\hat{\Psi}, \hat{\Phi}, \hat{\theta}) = \text{Argmax}_{\Psi, \Phi, \theta} \sum_{t=r+1}^{T-s} \ln g[(1 - \psi L^{-1})(1 - \phi L)y_t; \theta], \]

where \( g[.; \theta] \) denotes the Cauchy density function of \( \varepsilon_t \).

\[ g(\varepsilon) = \frac{1}{\pi} \left[ \frac{\theta}{(\varepsilon^2 + \theta^2)} \right] \]

(Jacobian = 1, even in MAR(1,1))
A MAR(1,1) model with Cauchy errors is estimated, using the AML method [Lanne, Saikkonen (2011)]:

$$(1 - 0.517L)(1 - 0.719L^{-1})y_t = \varepsilon_t,$$

where $\varepsilon_t$ is Cauchy distributed with location 0 and scale coefficient 1.89.

The spike chosen for the analysis is located close to the end of the trajectory to ensure that the predictive densities are computed from sufficiently many prior fitted values of the latent component $\hat{u}_t = y_t - \hat{\phi}y_{t-1}$. The inception of this bubble is around observation 144.
Univariate Noncausal Process
Forecasting from MAR(1,1)
Application: Bubble Analysis

Bitcoin/US Dollar exchange rate - Detrended
Bubble Analysis in Noncausal Processes

Bubble Outset

\[
y_{142} = 28.46, \quad y_{141} = 84.79
\]

\[
y_{143} = 30.65, \quad y_{142} = 28.46
\]
Univariate Noncausal Process
Forecasting from MAR(1,1)
Application : Bubble Analysis

Bubble Burst

\[ y_{144} = 222.33, y_{143} = 30.65 \]

\[ y_{145} = 126.74, y_{144} = 222.33 \]
The patterns: diamonds represent the standard forms of bivariate density of $y_{T+1}, y_{T+2}$ before and after the bubble. The three patterns in the middle depict various deformations that can be observed. These deformations concern the position of the density, which rotates, its shape that becomes stretched and/or disintegrated and the appearance of multiple modes, marked by dots in the diagram below.
References:

Simulations: \( Y_t = 0.1 Y_{t+1} + \varepsilon_t, \) \((\varepsilon_t) \text{i.i.d.} \sim C(0, 1)\)
Simulations: \( Y_t = 0.5 Y_{t+1} + \epsilon_t \), \((\epsilon_t) \text{i.i.d.} \sim \mathcal{N}(0, 1)\)
Simulations: $Y_t = 0.9 Y_{t+1} + \varepsilon_t,$ \hspace{1cm} $(\varepsilon_t)$ i.i.d. $\sim \mathcal{N}(0, 1)$
point forecast $\hat{y}_{T+1}$ mode of the estimated predictive density

$$\hat{y}_{T+1} = \text{mode}[^\pi(y_{T+1}|y_T, \hat{u}_T)].$$

The mode is a nonparametric predictor that exists for any distribution including the Cauchy.

The forecast error associated with the point forecast is:

$$fe_{T+1} = y_{T+1} - \hat{y}_{T+1},$$

i.e. the difference between the true and predicted $y_{T+1}$.

A robust interval forecast is

$$P[y_{T+1} \in (Q_T(\alpha/2), Q_T(1-\alpha/2)) = 1 - \alpha,$$

where $Q_T(.)$ is the quantile function of $^\pi(y_{T+1}|y_T, \hat{u}_T)$. The probability level $\alpha$ can be set depending on the desired coverage.
Univariate Noncausal Process
Forecasting from MAR(1,1)
Application: Bubble Analysis

Point Prediction for BITCOIN up to H=7 in one step,
\( \hat{\phi} = 0.71, \hat{\psi} = 0.67 \)
Univariate Noncausal Process
Forecasting from MAR(1,1)
Application: Bubble Analysis

Rolling forecast, lines: solid: MAR(1,1) process, dashed: forecast, dotted: prediction interval
Predictive density at horizon $H$

For $s = 1$ the conditional p.d.f. by the Bayes theorem can be written as:

$$
I(u_{T+1}, \ldots, u_{T+H} | u_T) = \frac{I(u_T, u_{T+1}, \ldots, u_{T+H})}{l_1(u_T)}
$$

$$
= I(u_T, \ldots, u_{T+H-1} | u_{T+H}) \frac{l_1(u_{T+H})}{l_1(u_T)}
$$

$$
= A \times \frac{B}{C}
$$

(8)

where $l_1$ denotes the stationary density of component $u$. 

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Bubble Analysis in Noncausal Processes
estimate A : Given that $u_t - \psi u_{t+1} = \varepsilon_t$ and $\varepsilon_t \sim g(\varepsilon_t)$, component A is:

$$l(u_T, \ldots, u_{T+H-1}|u_{T+H})$$

$$= l(u_T|u_{T+1})l(u_{T+1}|u_{T+2}) \ldots l(u_{T+H-1}|u_{T+H})$$

$$= g(u_T - \psi u_{T+1})g(u_{T+1} - \psi u_{T+2}) \ldots g(u_{T+H-1} - \psi u_{T+H}).$$

It can be estimated using the estimated parametric density $g$ of error $\varepsilon_t$, and the filtered $\hat{u}_T$ computed using the estimated $\Phi$ and $\Psi$ as follows:

$$\hat{g}(\hat{u}_T - \hat{\psi} u_{T+1})\hat{g}(u_{T+1} - \hat{\psi} u_{T+2}) \ldots \hat{g}(u_{T+H-1} - \hat{\psi} u_{T+H})$$
Functional estimator of predictive density

Estimate B and C: from the stationary densities:

\[ l_1(u) = \frac{1}{T} \sum_{t=1}^{T} \hat{g}(u - \hat{\psi} \hat{u}_t) \]

The predictive density estimator \( \hat{\Pi} \) at horizon \( H \), for \( s = 1 \) is:

\[
\hat{\Pi}(u_{T+1}, \ldots, u_{T+H} | \hat{u}_T) \\
\equiv \hat{g}(\hat{u}_T - \hat{\psi} u_{T+1}) \hat{g}(u_{T+1} - \hat{\psi} u_{T+2}) \cdots \hat{g}(u_{T+H-1} - \hat{\psi} u_{T+H}) \\
\times \frac{1}{T} \sum_{t=1}^{T} \hat{g}(u_{T+H} - \hat{\psi} \hat{u}_t) \\
\times \frac{1}{T} \sum_{t=1}^{T} \hat{g}(\hat{u}_T - \hat{\psi} \hat{u}_t) 
\]

(9)
Prediction of future $y$

- **Step 1**: Use data $(y_1, \ldots, y_T)$ to compute the filtered values of in-sample latent components $u$:
  $$\hat{\varepsilon}_2, \ldots, \hat{\varepsilon}_{T-1}, \hat{v}_1, \ldots, \hat{v}_{T-1}, \hat{u}_2, \ldots, \hat{u}_T.$$

- **Step 2**: Compute the predictive density $\hat{\Pi}$.

- **Step 3**: Use the Sampling-Importance-Resampling (SIR) method to simulate future $u$’s:
  $$u^s_{T+1}, \ldots, u^s_{T+H}.$$

- **Step 4**: Use the recursive formulas 1-3 to compute the future values:
  $$\hat{y}_{T+1}, \ldots, \hat{y}_{T+H}, \hat{\varepsilon}_T, \ldots, \hat{\varepsilon}_{T+H-1}, \hat{v}_T, \ldots, \hat{v}_{T+H-1}.$$