A New Class of Tests of Contagion with Applications

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Abstract

A new class of tests of contagion is proposed which identifies transmission channels of financial market crises through changes in higher order moments of the distribution of returns such as coskewness. Applying the framework to test for contagion in real estate and equity markets following the Hong Kong crisis in 1997-1998 and the US subprime mortgage crisis in 2007 shows that the coskewness based tests of contagion detect additional channels that are not identified by the correlation based tests. Implications of contagion in pricing exchange options where there is a change in higher order comoments of returns on the underlying assets, are also investigated.

Keywords: Contagion testing, coskewness, Lagrange multiplier tests, Hong Kong crisis, subprime mortgage crisis, exchange options.

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1 Introduction

A common empirical characteristic of financial crises is that asset return volatility increases during a crisis while average returns fall. This phenomenon is highlighted in the top panel of Table 1 which shows average daily returns on a Hong Kong securitized real estate index decreasing from 0.08% prior to the Hong Kong crisis in October 1997, to −0.45% during the crisis, whilst daily volatility increases respectively from 1.94%\(^2\) to 15.47%\(^2\). Table 1 shows that similar results occur for Hong Kong equities for the same period. From a mean-variance perspective where risk-averse agents need to be compensated for higher risk with higher average returns, this empirical phenomenon represents a potential puzzle.

Another feature of the asset returns in Table 1 is that the distribution of asset returns during the financial crises switch from negative skewness to positive skewness. As risk averse agents prefer positive skewness to negative skewness (Ingersoll 1990; Harvey and Siddique 2000), the lower average returns occurring in financial crises is at least partly explained by agents trading-off lower average returns for positive skewness. Thus, even though the mean and the variance of returns may appear negatively related, this ignores the role of skewness as an important third factor in understanding risk.

The importance of identifying the role of skewness and higher order moments in univariate distributions of asset returns during financial crises also applies to identifying the importance of comovements amongst moments in multivariate distributions, such as coskewness for example. The bottom panel of Table 1 shows that coskewness in asset returns between real estate and equity markets during the Hong Kong crisis switches from being negative prior to the crisis to being positive during the crisis period. This result is the bivariate analogue of the univariate skewness result. Again, the shift from negative coskewness to positive coskewness during the financial crisis is consistent with a reduction in average returns for risk averse agents in the presence of higher volatility.

The identification of significant changes in the moments of the distribution of asset returns forms the basis of recent tests of contagion. Here contagion is defined as changes in the moments of the distribution during a financial crisis over and above changes due to market fundamentals (Dornbusch, Park and Claessens 2000; Pericoli and Sbracia 2003). For a review of tests of contagion see Dungey, Fry, González-Hermosillo and Martin (2005). The earliest work is by King and Wadhwani (1990) who focus on
Table 1:

Descriptive statistics of daily securitized real estate and equity returns for Hong Kong during the Hong Kong crisis, expressed as a percentage: precrisis (January 1, 1996 to October 16, 1997); crisis (October 17, 1997 to June 30, 1998).

<table>
<thead>
<tr>
<th>Market</th>
<th>Precrisis Mean</th>
<th>Crisis</th>
<th>Precrisis Variance</th>
<th>Crisis</th>
<th>Precrisis Skewness</th>
<th>Crisis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Real Estate</td>
<td>0.08</td>
<td>-0.45</td>
<td>1.94</td>
<td>15.47</td>
<td>-0.69</td>
<td>0.64</td>
</tr>
<tr>
<td>Equity</td>
<td>0.07</td>
<td>-0.16</td>
<td>1.82</td>
<td>10.44</td>
<td>-0.52</td>
<td>0.18</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Correlation</th>
<th>Coskewness1(^{(a)})</th>
<th>Coskewness2(^{(b)})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Real Estate - Equity</td>
<td>0.84</td>
<td>0.90</td>
<td>-0.51</td>
</tr>
</tbody>
</table>

(a) Coskewness1: Coskewness between real estate returns and squared equity returns.
(b) Coskewness2: Coskewness between squared real estate returns and equity returns.

Changes in the correlation between precrisis and crisis periods. See Table 1 where the correlation between returns in real estate and equity markets in Hong Kong increases from 0.84 prior to the crisis, to 0.90 during the crisis. The correlation tests of contagion interpret an increase in correlations as providing evidence of contagion as it represents additional comovements in asset returns during the crisis period not present in the precrisis period. Forbes and Rigobon (2002) augment this test by recognizing the need to condition on the increase in volatility of asset returns in the asset market that is the source of the crisis when computing correlations, otherwise any observed increases in correlations may be purely spurious.

While focusing on correlations is a natural starting point in testing for contagion, the descriptive statistics in Table 1 also highlight the possibility of additional contagious channels operating through higher order comoments. The aim of this paper is to develop a new class of tests of contagion which focus on testing for changes in higher order comoments. To motivate the form of the newly proposed tests of contagion, an asset pricing model based on the stochastic discount factor model is developed in Section 2 which extends the models of Kraus and Litzenberger (1976), Lee, Moy and Lee (1996), and more recently Harvey and Siddique (2000). An important feature of the asset pricing model is that it provides an explicit expression for pricing risk in terms of risk prices and risk quantities where the latter are represented by higher
order conditional moments of the distribution of asset returns. In defining the conditional moments it is shown how the form of the conditioning set is fundamental to the construction of valid tests of contagion. In this context the model proposed here is related to the recent work of Guidolin and Timmermann (2007) where the conditional moments are expressed in terms of the states of nature.

The new tests of contagion are presented in Section 3. Whilst the focus is on testing for changes in coskewness, the testing framework developed is general enough whereby other tests of contagion based on alternative higher order moments, including cokurtosis, follow. An important special case of the proposed framework is that it nests the correlation based tests of contagion of Forbes and Rigobon (2002). The generality of this framework also lends itself to extending the existing tests of multivariate normality proposed by Mardia (1970), Bera and John (1983) and Richardson and Smith (1993). Some of these extensions are explored by Lye and Martin (1994) in a univariate context, which follow from extending the univariate class of generalized exponential distributions developed by Cobb, Koppstein and Chen (1983) and Lye and Martin (1993), to a bivariate class of distributions.

The new tests of contagion are applied to two financial crises in Section 4. Unlike most of the previous empirical applications of contagion tests, the focus here is on real estate markets which played an important role in both crises. The first is the Hong Kong crisis in October of 1997 and the second is the more recent US based subprime crisis beginning July of 2007. The empirical results presented reveal additional channels of contagion that are not identified by the correlation based tests. To explore the effects of changes in coskewness on asset pricing during financial crises further, the implications of changes in coskewness on exchange options are investigated in Section 5. Based on the parameterizations chosen in the simulation experiments, the results show that the Black-Scholes model of exchange options underprices by around 25% when there is positive coskewness in returns during financial crises. The paper concludes in Section 6 with a summary of the main results together with some suggestions for future research. Data sources and key derivations of the test statistics are presented in two appendices.
2 A Portfolio Model of Higher Order Moments

The descriptive statistics presented in Table 1 highlight the importance of modelling risk during financial crises using higher order moments. To formalize the relationship between risk and the moments of the distribution of returns, a portfolio model is now presented which builds on the work of Kraus and Litzenberger (1976), Lee et al. (1996) and Harvey and Siddique (2000). This model represents an extension of the mean-variance framework where the expected excess return on assets is expressed in terms of risk prices which are a function of the risk preferences of investors, and risk quantities which are a function of higher order conditional moments including skewness and coskewness. From the perspective of modelling and testing for contagion, the form of the conditioning set used to define the risk quantities is shown to be fundamental to the analysis.

Consider the equilibrium pricing equation

\[ E \left[ m_{t+1} (1 + r_{i,t+1}) \mid s_{t+1} \right] = 1, \]  

where \( m_{t+1} \) is the stochastic discount factor, \( r_{i,t+1} \) is the return on asset \( i \), and \( E \left[ \cdot \right] \) is the conditional expectations operator based on information at time \( t \), suppressed here for ease of notation, and the states of nature at time \( t + 1 \) given by \( s_{t+1} \). By defining

\[ \mu_{i,t+1} = E \left[ r_{i,t+1} \mid s_{t+1} \right], \]

as the conditional expectation of returns at time \( t + 1 \), the equilibrium pricing equation in (1) is rewritten as

\[ E \left[ m_{t+1} (r_{i,t+1} - \mu_{i,t+1}) \mid s_{t+1} \right] = 0. \]

From intertemporal asset pricing theory, \( m_{t+1} \) is given by the Euler equation

\[ m_{t+1} = \frac{\beta u' (w_{t+1})}{u' (w_t)}, \]

where \( \beta \) is the time discount rate and \( u' (\cdot) \) is the marginal utility obtained from wealth, \( w \). The relationship between wealth at time \( t \) and \( t+1 \) is given by the wealth generating equation

\[ w_{t+1} = (1 + r_{p,t+1}) w_t, \]
where $r_{p,t+1}$ is the return on the portfolio. For a portfolio containing $N$ assets with returns $(r_{1,t+1}, r_{2,t+1}, \cdots, r_{N,t+1})$, $r_{p,t+1}$ is given by

$$r_{p,t+1} = \sum_{i=1}^{N} \alpha_i r_{i,t+1},$$

(6)

where $\alpha_i$ is the weight of the $i^{th}$ asset in the portfolio with the short sale restriction, $\alpha_i \geq 0$, and adding-up restriction, $\sum_{i=1}^{N} \alpha_i = 1$.

To develop an asset pricing model containing higher-order moments in returns, rewrite $u'(w_{t+1})$ in (4) by using equations (5) and (6)

$$u'(w_{t+1}) = u' \left( \left( 1 + \sum_{i=1}^{N} \alpha_i r_{i,t+1} \right) w_t \right).$$

(7)

This expression can be approximated by using a Taylor series expansion around the conditional expected returns $\mu_{i,t+1}$ in (2). In the case of a bivariate portfolio model, $N = 2$ in (6), the expansion takes the form

$$u'(w_{t+1}) \simeq u'(w_t + w_t \mu_{p,t+1})$$

$$+ u''(w_t + w_t \mu_{p,t+1}) \alpha_1 w_t (r_{1,t+1} - \mu_{1,t+1})$$

$$+ u''(w_t + w_t \mu_{p,t+1}) \alpha_2 w_t (r_{2,t+1} - \mu_{2,t+1})$$

$$+ \frac{1}{2} u'''(w_t + w_t \mu_{p,t+1}) \alpha_1^2 w_t^2 (r_{1,t+1} - \mu_{1,t+1})^2$$

$$+ \frac{1}{2} u'''(w_t + w_t \mu_{p,t+1}) \alpha_2^2 w_t^2 (r_{2,t+1} - \mu_{2,t+1})^2$$

$$+ \frac{1}{2} u'''(w_t + w_t \mu_{p,t+1}) \alpha_1 \alpha_2 w_t^2 (r_{1,t+1} - \mu_{1,t+1}) (r_{2,t+1} - \mu_{2,t+1})$$

$$+ o(w_t),$$

where $o(w_t)$ represents the higher order terms in the Taylor series expansion, and $\mu_{p,t+1}$ is the conditional expected return on the portfolio defined as

$$\mu_{p,t+1} = E \left[ \sum_{i=1}^{N} \alpha_i r_{i,t+1} \right] = \sum_{i=1}^{N} \alpha_i \mu_{i,t+1}.$$
From the definition of $m_{t+1}$ in (4)

\[
m_{t+1} = \frac{\beta}{w'(w_t)} \left\{ w' (w_t + w_t \mu_{p,t+1})
+ w'' (w_t + w_t \mu_{p,t+1}) \alpha_1 w_t (r_{1,t+1} - \mu_{1,t+1})
+ w'' (w_t + w_t \mu_{p,t+1}) \alpha_2 w_t (r_{2,t+1} - \mu_{2,t+1})
+ \frac{1}{2} w'' (w_t + w_t \mu_{p,t+1}) \alpha_1^2 w_t^2 (r_{1,t+1} - \mu_{1,t+1})^2
+ \frac{1}{2} w'' (w_t + w_t \mu_{p,t+1}) \alpha_2^2 w_t^2 (r_{2,t+1} - \mu_{2,t+1})^2
+ \frac{1}{2} w'' (w_t + w_t \mu_{p,t+1}) \alpha_1 \alpha_2 w_t^2 (r_{1,t+1} - \mu_{1,t+1}) (r_{2,t+1} - \mu_{2,t+1}) \right\} + o(w_t).
\]

(9)

From (3), $E [m_{t+1} (r_{i,t+1} - \mu_{i,t+1}) | s_{t+1}] = E [m_{t+1} r_{i,t+1} | s_{t+1}] - E [m_{t+1} | s_{t+1}] \mu_{i,t+1}$, so

\[
E [m_{t+1} r_{i,t+1} | s_{t+1}] = E [m_{t+1} (r_{i,t+1} - \mu_{i,t+1}) | s_{t+1}] + E [m_{t+1} | s_{t+1}] \mu_{i,t+1}.
\]

(10)

Also from (2), $E [m_{t+1} (1 + r_{i,t+1}) | s_{t+1}] = E [m_{t+1} | s_{t+1}] + E [m_{t+1} r_{i,t+1} | s_{t+1}] = 1$, in which case

\[
E [m_{t+1} | s_{t+1}] + E [m_{t+1} (r_{i,t+1} - \mu_{i,t+1}) | s_{t+1}] + E [m_{t+1} | s_{t+1}] \mu_{i,t+1} = 1,
\]

or

\[
1 + \frac{E [m_{t+1} (r_{i,t+1} - \mu_{i,t+1}) | s_{t+1}] + \mu_{i,t+1}}{E [m_{t+1} | s_{t+1}]} = \frac{1}{E [m_{t+1} | s_{t+1}]}.
\]

(11)

Using the result (Campbell, Lo and MacKinlay 1997, p. 294)

\[
E [m_{t+1} | s_{t+1}] = \frac{1}{1 + r_{f,t+1}},
\]

where $r_{f,t+1}$ is the risk free rate of interest, (11) becomes

\[
(1 + r_{f,t+1}) E [m_{t+1} (r_{i,t+1} - \mu_{i,t+1}) | s_{t+1}] + \mu_{i,t+1} = r_{f,t+1}.
\]

Upon rearranging

\[
\mu_{i,t+1} - r_{f,t+1} = -(1 + r_{f,t+1}) E [m_{t+1} (r_{i,t+1} - \mu_{i,t+1})],
\]

and using the expression for $m_{t+1}$ in (9) with higher order terms suppressed for simplicity, gives
\[ \mu_{i,t+1} - r_{f,t+1} = \phi_1 E \left[ (r_{1,t+1} - \mu_{1,t+1}) (r_{i,t+1} - \mu_{i,t+1}) \mid s_{t+1} \right] \\
+ \phi_2 E \left[ (r_{2,t+1} - \mu_{2,t+1}) (r_{i,t+1} - \mu_{i,t+1}) \mid s_{t+1} \right] \\
+ \phi_3 E \left[ (r_{1,t+1} - \mu_{1,t+1})^2 (r_{i,t+1} - \mu_{i,t+1}) \mid s_{t+1} \right] \\
+ \phi_4 E \left[ (r_{2,t+1} - \mu_{2,t+1})^2 (r_{i,t+1} - \mu_{i,t+1}) \mid s_{t+1} \right] \\
+ \phi_5 E \left[ (r_{1,t+1} - \mu_{1,t+1}) (r_{2,t+1} - \mu_{2,t+1}) (r_{i,t+1} - \mu_{i,t+1}) \mid s_{t+1} \right], \tag{12} \]

where

\[
\phi_1 = -\frac{\beta u'' (w_t + w_t \mu_{p,t+1}) (1 + r_{f,t+1}) \alpha_1 w_t}{u'(w_t)} \\
\phi_2 = -\frac{\beta u'' (w_t + w_t \mu_{p,t+1}) (1 + r_{f,t+1}) \alpha_2 w_t}{u'(w_t)} \\
\phi_3 = -\frac{\beta u'' (w_t + w_t \mu_{p,t+1}) (1 + r_{f,t+1}) \alpha_1^2 w_t^2}{2 u'(w_t)} \\
\phi_4 = -\frac{\beta u'' (w_t + w_t \mu_{p,t+1}) (1 + r_{f,t+1}) \alpha_2^2 w_t^2}{2 u'(w_t)} \\
\phi_5 = -\frac{\beta u'' (w_t + w_t \mu_{p,t+1}) (1 + r_{f,t+1}) \alpha_1 \alpha_2 w_t^2}{2 u'(w_t)}. \tag{13} \]

Equation (12) gives the equilibrium pricing equation in terms of second and higher-order moments, including comoments, which is valid for \( \forall i \). In the case of asset 1, the expected excess return is given by setting \( i = 1 \) in (12)

\[ \mu_{1,t+1} - r_{f,t+1} = \phi_1 E \left[ (r_{1,t+1} - \mu_{1,t+1})^2 \mid s_{t+1} \right] \\
+ \phi_2 E \left[ (r_{2,t+1} - \mu_{2,t+1}) (r_{1,t+1} - \mu_{1,t+1}) \mid s_{t+1} \right] \\
+ \phi_3 E \left[ (r_{1,t+1} - \mu_{1,t+1})^3 \mid s_{t+1} \right] \\
+ \phi_4 E \left[ (r_{2,t+1} - \mu_{2,t+1})^2 (r_{1,t+1} - \mu_{1,t+1}) \mid s_{t+1} \right] \\
+ \phi_5 E \left[ (r_{1,t+1} - \mu_{1,t+1})^2 (r_{2,t+1} - \mu_{2,t+1}) \mid s_{t+1} \right]. \tag{14} \]

This equation shows that the risk of an asset is decomposed in terms of risk prices, as given by \( \phi_i \), and risk quantities, as given by the conditional expectations. The risk
prices are expressed in terms of the various risk aversion measures arising from the utility function of the investor, the discount parameter $\beta$, the share of the asset in the portfolio $\alpha_i$, and the risk free interest rate $r_f$. In the case of $\phi_1$ and $\phi_2$, the risk prices are a function of the relative risk aversion parameter $-u''(w_t + w_t\mu_{p,t+1}) w_t/w_t'$. The risk quantities contain the usual second moment terms, the variance and covariance, as well as the third order moments, skewness $E\left[(r_{2,t+1} - \mu_{2,t+1})^3 \big| s_{t+1}\right]$, and coskewness, $E\left[(r_{1,t+1} - \mu_{1,t+1})^2 (r_{2,t+1} - \mu_{2,t+1}) \big| s_{t+1}\right]$, $E\left[(r_{1,t+1} - \mu_{1,t+1}) (r_{2,t+1} - \mu_{2,t+1})^2 \big| s_{t+1}\right]$. An alternative approach that also shows the importance of skewness in modelling contagion is by Yuan (2005).

To highlight the properties of this model, the risk-return trade-off surface between the conditional mean (excess), conditional standard deviation and the conditional skewness, is presented in Figure 1. The parameters in (14) are chosen as

$$\phi_1 = 0.5, \phi_2 = 2.0, \phi_3 = \phi_4 = \phi_5 - 1.5,$$

with the conditional covariance set at $E\left[(r_{2,t+1} - \mu_{2,t+1}) (r_{1,t+1} - \mu_{1,t+1}) \big| s_{t+1}\right] = 0.8$, and conditional coskewness at

$$E\left[(r_{2,t+1} - \mu_{2,t+1})^2 (r_{1,t+1} - \mu_{1,t+1}) \big| s_{t+1}\right] = E\left[(r_{1,t+1} - \mu_{1,t+1})^2 (r_{2,t+1} - \mu_{2,t+1}) \big| s_{t+1}\right] = 0.$$

The effects of a crisis on the risk-return trade-off is highlighted in Figure 2 where conditional skewness changes from $-1$ in the precrisis period to $+1$ in the crisis period. In the precrisis period for asset 1, the conditional standard deviation is set at

$$\sqrt{E\left[(r_{1,t+1} - \mu_{1,t+1})^2 \big| s_{t+1}\right] = \text{Precrisis}} = 1,$$

which corresponds to a conditional mean of $E\left[r_{1,t+1} \big| s_{t+1} = \text{Precrisis}\right] = 3.6$. In the crisis period the conditional standard deviation doubles to

$$\sqrt{E\left[(r_{1,t+1} - \mu_{1})^2 \big| s_{t+1} = \text{Crisis}\right] = 2},$$

while the conditional mean falls to $E\left[r_{1,t+1} \big| s_{t+1} = \text{Crisis}\right] = 2.1$.

The conditional expectations that govern the risk quantities are a function of three sources of information that are fundamental to modelling contagion. The first is information at time $t$ that determines the market fundamentals which drive asset returns. As contagion represents the additional comovements in asset returns over and above
Figure 1: Mean - standard deviation - skewness tradeoffs.

Figure 2: Risk - return tradeoff between conditional mean (excess) and conditional standard deviation for a precrisis period (negative skewness of -1) and a crisis period (positive skewness of +1).
that predicted by changes in market fundamentals, the identification of contagion requires the extraction of market fundamentals from the returns series. In implementing the model in the empirical application below, the market fundamentals are formally modelled by specifying a VAR which includes among other things, variables that contribute to the common factors that determine asset returns. Second, it is important to condition on the asset market that is the source of the crisis, otherwise quantities based on correlations for example, can result in spurious inferences. In the context of testing for contagion, this point is first noted by Forbes and Rigobon (2002) who show that increases in volatility of the source asset market can lead to increases in correlation between two markets even when there is no change in the relationship between the two markets. Third, a financial crisis is identified by a change in the probabilities of the states of nature $s_{t+1}$. In the development of the contagion tests below, the approach is to treat the changes in the states of nature as a discrete structural break whereby moments are computed for a precrisis period and a crisis period. This interpretation of contagion tests is discussed in greater detail by Dungey et al. (2005), who show that many of the test of contagion can be viewed as tests of structural breaks. An alternative approach adopted by Guidolin and Timmermann (2007), is to specify a Markovian switching model.

### 3 Contagion Testing of Higher Order Moments

#### 3.1 Generalized Normality

The portfolio model presented in the previous section demonstrates the importance of higher order comoments of returns in understanding the effects of contagion on risk during financial crises. To develop new tests of contagion through higher order moments it is necessary to specify a nonnormal multivariate returns distribution. The approach adopted here is to extend the existing univariate class of generalized exponential distributions developed by Cobb et al. (1983) and Lye and Martin (1993), to multivariate distributions. Define $r = \{r_1, r_2, \cdots, r_K\}$ as a $K$ dimensional vector of random variables. The multivariate generalized exponential family of distributions is given by

$$f(r) = \exp \left( \sum_{i=1}^{M} \theta_i g_i(r) - \eta \right),$$

(15)
where \( \theta = \{ \theta_1, \theta_2, \cdots, \theta_M \} \) is a \( M \) dimensional vector of parameters, \( g_i (r) \) is an arbitrary function of the \( K \) dimensional random variable \( r \), and \( \eta \) is a normalizing constant defined as

\[
\eta = \ln \int \cdots \int \exp \left( \sum_{i=1}^{M} \theta_i g_i (r) \right) dr_1 dr_2 \cdots dr_K,
\]

which ensures that \( f (r) \) is a well defined probability distribution with the property that

\[
\int \cdots \int \exp \left( \sum_{i=1}^{M} \theta_i g_i (r) - \eta \right) dr_1 dr_2 \cdots dr_K = 1.
\]

Typical choices for \( g_i (r) \) in (15) are polynomials and cross-products in the elements of \( r \). For example, setting \( g_1 (r) = r_1^2 \) yields

\[
f (r) = \exp \left( \sum_{i=1}^{M} \theta_i r_i^2 - \eta \right),
\]

which is the \( M \) dimensional independent normal distribution. A bivariate example (\( K = 2 \)) is given by setting \( M = 3 \) and choosing \( g_1 (r) = r_1^2 \), \( g_2 (r) = r_2^2 \), and \( g_3 (r) = r_1 r_2 \). This yields the bivariate normal distribution

\[
f (r) = \exp \left( \theta_1 r_1^2 + \theta_2 r_2^2 + \theta_3 r_1 r_2 - \eta \right),
\]

for the case of zero means, where \( \theta_1 \) and \( \theta_2 \) control the respective variances and \( \theta_3 \) controls the degree of dependence which is a function of the correlation coefficient. A natural generalization of (18) which allows for higher order moments is given by the following bivariate generalized normal distribution with \( M = 6 \) in (15)

\[
f (r) = \exp \left( \theta_1 r_1^2 + \theta_2 r_2^2 + \theta_3 r_1 r_2 + \theta_4 r_1^2 r_2^2 + \theta_5 r_1^2 r_2 + \theta_6 r_1^2 r_2^2 - \eta \right).
\]

The terms \( r_1^2 r_2 \) and \( r_1^2 r_2^2 \) represent two measures of coskewness between \( r_1 \) and \( r_2 \) which are controlled by the parameters \( \theta_4 \) and \( \theta_5 \) respectively, while the term \( r_1^2 r_2^2 \) represents cokurtosis between \( r_1 \) and \( r_2 \), which is controlled by the parameter \( \theta_6 \). For this distribution to have finite moments of all orders, it is required that \( \theta_6 < 0 \). This distribution allows for three levels of dependence amongst asset returns. The first is the usual channel of contagion which is controlled by the parameter \( \theta_3 \), effectively the correlation parameter. The second channel is through the parameters \( \theta_4 \) and \( \theta_5 \) where dependence is captured by the interaction between the first moment of asset 1 (\( r_1 \)) and
the second moment of asset 2 \( (r_2^2) \), namely coskewness. The third channel is controlled by the cokurtosis parameter \( \theta_6 \), which allows for the interaction of the second moments of the two random variables.

As the bivariate generalized normal distribution in (19) nests the bivariate normal distribution as a special case by imposing the restrictions \( \theta_4 = \theta_5 = \theta_6 = 0 \), a convenient testing framework is to develop Lagrange multiplier tests of contagion. Moreover, as the multivariate normal distribution is nested within the generalized family of distributions, this suggests that the multivariate tests of normality proposed by Mardia (1970), Bera and John (1983) and Richardson and Smith (1993), can also be generalized. In developing tests of contagion and multivariate normality in general, the following theorem provides a convenient method for computing the pertinent standard errors.

**Theorem 1** Let \( r \) be an iid random variable of dimension \( K \) with the generalized exponential distribution

\[
    f(r) = \exp(h - \eta),
\]

where \( h = \sum_{i=1}^{M} \theta_i g_i(r) \), \( \theta \) is a \( M \) vector of parameters summarizing the moments of the distribution and \( \eta \) is the normalizing constant defined in (16). The information matrix is given by

\[
    I = T \left( E \left[ \frac{\partial h}{\partial \theta} \frac{\partial h}{\partial \theta'} \right] - E \left[ \frac{\partial h}{\partial \theta} \right] E \left[ \frac{\partial h}{\partial \theta'} \right] \right),
\]

where \( \ln L_t = \ln f(r_t) \) represents the log of the likelihood at the \( t^{th} \) observation and \( T \) is the sample size.

**Proof.** From equation (20) the log of the likelihood at observation \( t \) is

\[
    \ln L_t = h_t - \eta,
\]

where \( h_t = \sum_{i=1}^{M} \theta_i g_i(r_t) \), while the first and second derivatives are respectively

\[
    \frac{\partial \ln L_t}{\partial \theta} = \frac{\partial h_t}{\partial \theta} - \frac{\partial \eta}{\partial \theta'},
\]

\[
    \frac{\partial^2 \ln L_t}{\partial \theta \partial \theta'} = \frac{\partial^2 h_t}{\partial \theta \partial \theta'} - \frac{\partial^2 \eta}{\partial \theta \partial \theta'}.
\]
Taking expectations of the second derivative and changing the sign yields the information matrix at $t$

$$I_t = -E \left[ \frac{\partial^2 \ln L_t}{\partial \theta \partial \theta'} \right]$$

$$= \frac{\partial^2 \eta}{\partial \theta \partial \theta'} - E \left[ \frac{\partial^2 h_t}{\partial \theta \partial \theta'} \right]. \quad (25)$$

This expression is simplified by differentiating

$$\eta = \ln \int \exp (h) \, dr,$$

which gives

$$\frac{\partial \eta}{\partial \theta} = \frac{\int \frac{\partial h}{\partial \theta} \exp [h] \, dr}{\int \exp [h] \, dr} = E \left[ \frac{\partial h}{\partial \theta} \right]. \quad (26)$$

Differentiating a second time gives

$$\frac{\partial^2 \eta}{\partial \theta \partial \theta'} = \left( \int \frac{\partial^2 h}{\partial \theta \partial \theta'} \exp [h] \, dr + \int \frac{\partial h}{\partial \theta} \frac{\partial h}{\partial \theta'} \exp [h] \, dr \right) \left( \int \exp [h] \, dr \right)^2$$

$$- \left( \int \frac{\partial h}{\partial \theta} \exp [h] \, dr \right) \left( \int \frac{\partial h}{\partial \theta'} \exp [h] \, dr \right) \left( \int \exp [h] \, dr \right)^2$$

$$= E \left[ \frac{\partial^2 h}{\partial \theta \partial \theta'} \right] + E \left[ \frac{\partial h}{\partial \theta} \frac{\partial h}{\partial \theta'} \right] - E \left[ \frac{\partial h}{\partial \theta} \right] E \left[ \frac{\partial h}{\partial \theta'} \right]. \quad (27)$$

Substituting this expression into the information matrix in (25) yields

$$I_t = E \left[ \frac{\partial^2 h}{\partial \theta \partial \theta'} \right] + E \left[ \frac{\partial h}{\partial \theta} \frac{\partial h}{\partial \theta'} \right] - E \left[ \frac{\partial h}{\partial \theta} \right] E \left[ \frac{\partial h}{\partial \theta'} \right] - E \left[ \frac{\partial^2 h_t}{\partial \theta \partial \theta'} \right]$$

$$= E \left[ \frac{\partial h}{\partial \theta} \frac{\partial h}{\partial \theta'} \right] - E \left[ \frac{\partial h}{\partial \theta} \right] E \left[ \frac{\partial h}{\partial \theta'} \right], \quad (28)$$

where the last step uses the property that $r$ is iid, so $E \left[ \frac{\partial^2 h_t}{\partial \theta \partial \theta'} \right] = E \left[ \frac{\partial^2 h}{\partial \theta \partial \theta'} \right]$. Finally,

$$I = \sum_t I_t$$

$$= T \left( E \left[ \frac{\partial h}{\partial \theta} \frac{\partial h}{\partial \theta'} \right] - E \left[ \frac{\partial h}{\partial \theta} \right] E \left[ \frac{\partial h}{\partial \theta'} \right] \right),$$
from the iid assumption.

The advantage of (28) is that it is not necessary to use the normalizing constant \( \eta \), to derive the information matrix. Instead, the information matrix can be conveniently derived simply by taking expectations of the functions of the first derivatives of \( h \).

Having derived the information matrix, the Lagrange multiplier statistic is given by

\[
LM = G' I^{-1} G,
\]

where

\[
G = \left. \frac{\partial \ln L_t}{\partial \theta} \right|_{\theta = \theta_0},
\]

is the gradient vector of the log of the likelihood and

\[
I = \left. \frac{\partial^2 \ln L_t}{\partial \theta \partial \theta'} \right|_{\theta = \theta_0},
\]

is the information matrix, which are both evaluated under the null \( \theta = \theta_0 \).

As an example, consider the following generalization of the bivariate normal distribution in (18)

\[
f(r_{1,t}, r_{2,t}) = \exp \left[ -\frac{1}{2} \left( \frac{1}{1 - \rho^2} \right) \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 + \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 - 2\rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) + \phi \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 - \eta \right],
\]

where

\[
\eta = \ln \int \int \exp \left[ -\frac{1}{2} \left( \frac{1}{1 - \rho^2} \right) \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 + \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 - 2\rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) + \phi \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \right] dr_1 dr_2
\]

and

\[
h = -\frac{1}{2} \left( \frac{1}{1 - \rho^2} \right) \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 + \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 - 2\rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) + \phi \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2.
\]
A test of the restriction

$$H_0 : \phi = 0,$$

(35)

constitutes a test of coskewness. Under the null hypothesis, the distribution is bivariate normal where the maximum likelihood estimators of the unknown parameters are

$$\hat{\mu}_i = \frac{1}{T} \sum r_{i,t}; \hat{\sigma}_i^2 = \frac{1}{T} \sum (r_{i,t} - \hat{\mu}_i)^2, i = 1, 2; \hat{\rho} = \frac{1}{T} \sum \left( \frac{r_{1,t} - \hat{\mu}_1}{\hat{\sigma}_1} \right) \left( \frac{r_{2,t} - \hat{\mu}_2}{\hat{\sigma}_2} \right).$$

(36)

The Lagrange multiplier statistic is (see Appendix A.1 for details)

$$LM = \frac{T}{4\hat{\rho}^2 + 2} \left[ \frac{1}{T} \sum_{t=1}^T \left( \frac{r_{1,t} - \hat{\mu}_1}{\hat{\sigma}_1} \right) \left( \frac{r_{2,t} - \hat{\mu}_2}{\hat{\sigma}_2} \right)^2 \right]^2,$$

(37)

Under the null, $LM$ is distributed asymptotically as $LM \overset{d}{\rightarrow} \chi_1^2$.

### 3.2 Contagion Tests

In developing tests of contagion, two types are studied, although given the properties of the generalized multivariate normal distributions discussed above, clearly a wide range of comoments can be specified. The first is the Forbes and Rigobon (2002) contagion test which is based on testing changes in correlations, and the second is a contagion test based on changes in coskewness. In deriving the contagion tests the following notation is used. The precrisis period is denoted as $x$, and the crisis period as $y$. The correlation between the two asset returns is denoted as $\rho_x$ (precrisis) and $\rho_y$ (crisis). Finally, the sample sizes of the precrisis and crisis periods are respectively $T_x$ and $T_y$.

#### 3.2.1 Contagion Test Based on Changes in Correlations

Consider testing for contagion from a source asset market $i$, to a recipient, market $j$. The Forbes and Rigobon (2002) statistic ($FR$) to test for contagion from $i$ to $j$, is

$$FR(i \rightarrow j) = \frac{\left( \hat{\nu}_{y|x_i} - \hat{\rho}_x \right)}{\sqrt{Var(\hat{\nu}_{y|x_i} - \hat{\rho}_x)}}^2,$$

(38)

where

$$\hat{\nu}_{y|x_i} = \frac{\hat{\rho}_y}{\sqrt{1 + \left( \frac{s_{y,i}^2 - s_{x,i}^2}{s_{y,i}^2} \right) (1 - \hat{\rho}_y)}}$$

(39)
is the adjusted sample correlation coefficient where $s_{x,i}^2$ and $s_{y,i}^2$ are the sample variances of asset returns in market $i$ during the precrisis and crisis periods respectively. $\hat{\rho}_x$ and $\hat{\rho}_y$ are the sample correlation coefficients during the precrisis and crisis periods. The statistic $FR$, is actually the square of the statistic suggested by Forbes and Rigobon (2002), thereby making (38) a two-sided test. The variance is given by (see Appendix A.2 for details)

$$Var \left( \hat{\rho}_{y|x} - \hat{\rho}_x \right) = Var \left( \hat{\rho}_{y|x} \right) + Var \left( \hat{\rho}_x \right) - 2Cov \left( \hat{\rho}_{y|x}, \hat{\rho}_x \right),$$

(40)

where

$$Var \left( \hat{\rho}_{y|x} \right) = \frac{1}{T_y} \frac{(1 + \delta)^2}{1 + \delta (1 - \rho_y^2)} \left[ \frac{1}{T_y} \left( (2 - \rho_y^2) (1 - \rho_y^2)^2 \right) + \frac{1}{T_x} \left( \rho_y^2 (1 - \rho_y^2)^2 \right) \right],$$

$$Var \left( \hat{\rho}_x \right) = \frac{1}{T_x} (1 - \rho_x^2)^2,$$

(41)

and

$$Cov \left( \hat{\rho}_{y|x}, \hat{\rho}_x \right) = \frac{1}{2T_x} \frac{\rho_y \rho_x (1 - \rho_y^2) (1 - \rho_x^2) (1 + \delta)}{\sqrt{\left[ 1 + \delta (1 - \rho_y^2)^2 \right]^3}},$$

$$\delta = \left( \frac{\sigma_{y,i}^2 - \sigma_{x,i}^2}{\sigma_{x,i}^2} \right),$$

(42)

which is the proportionate change in the volatility of returns in the source asset market, market $i$. This statistic is computed by replacing the population quantities in (41) and (42) by their respective sample estimates.

An important special case of the variance expression in (40) occurs under the assumption of independence in both periods, $\rho_x = \rho_y = 0$, in which case (40) reduces to

$$Var \left( \hat{\rho}_{y|x} - \hat{\rho}_x \right) = \frac{1}{(1 + \delta) T_y} + \frac{1}{T_x}.$$

A further simplification is achieved under the additional assumption of no increase in the volatility in the source asset market ($\delta = 0$)

$$Var \left( \hat{\rho}_{y|x} - \hat{\rho}_x \right) = \frac{1}{T_y} + \frac{1}{T_x},$$

which is just the usual asymptotic formula for the variance of the difference in two independent sample correlation coefficients.

Under the null hypothesis of no contagion, the correlation test of contagion is asymptotically distributed as

$$FR \left( i \rightarrow j \right) \overset{d}{\longrightarrow} \chi^2_1.$$

(43)
Large values of the test statistic represent evidence of a significant change in the co-movement between asset returns during a crisis period.

### 3.2.2 Contagion Test Based on Changes in Coskewness

The coskewness test of contagion is based on identifying significant changes in coskewness between a crisis period and a precrisis period. Two variants of the test are specified depending on whether the asset market at the source of the crisis is expressed in terms of returns or squared returns in computing coskewness.

\[
CS_1 (i \rightarrow j; r_i^1, r_j^2) = \left( \frac{\hat{\psi}_y (r_i^1, r_j^2) - \hat{\psi}_x (r_i^1, r_j^2)}{\sqrt{\frac{4\hat{\nu}_{y|x} + 2}{T_y} + \frac{4\hat{\nu}_{x}^2 + 2}{T_x}}} \right)^2,
\]

\[
CS_2 (i \rightarrow j; r_i^2, r_j^1) = \left( \frac{\hat{\psi}_y (r_i^2, r_j^1) - \hat{\psi}_x (r_i^2, r_j^1)}{\sqrt{\frac{4\hat{\nu}_{y|x} + 2}{T_y} + \frac{4\hat{\nu}_{x}^2 + 2}{T_x}}} \right)^2,
\]

where

\[
\hat{\psi}_y (r_i^m, r_j^n) = \frac{1}{T_y} \sum_{t=1}^{T_y} \left( \frac{y_{i,t} - \hat{\mu}_{y_i}}{\hat{\sigma}_{y_i}} \right)^m \left( \frac{y_{j,t} - \hat{\mu}_{y_j}}{\hat{\sigma}_{y_j}} \right)^n,
\]

\[
\hat{\psi}_x (r_i^m, r_j^n) = \frac{1}{T_x} \sum_{t=1}^{T_x} \left( \frac{x_{i,t} - \hat{\mu}_{x_i}}{\hat{\sigma}_{x_i}} \right)^m \left( \frac{x_{j,t} - \hat{\mu}_{x_j}}{\hat{\sigma}_{x_j}} \right)^n.
\]

Under the null hypothesis of no contagion, the coskewness tests of contagion are asymptotically distributed as

\[
CS_1 (i \rightarrow j), CS_2 (i \rightarrow j) \xrightarrow{d} \chi^2_1.
\]

### 3.2.3 Finite Sample Properties

Given that precrisis sample periods tend to be relatively large, whereas crisis sample periods tend to be relatively short, the finite sample distribution properties of the contagion test statistics under the null hypothesis of no contagion, are investigated based on the following Monte Carlo experiments. Let the precrisis and crisis variance-covariance matrices of the returns on the two assets be given by respectively

\[
V_x = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad V_y = \begin{bmatrix} 10 & 9.682 \\ 9.682 & 15 \end{bmatrix},
\]
which yield correlations of $\rho_x = 0.5$ and $\rho_y = 0.791$. The increase in the volatility of returns between the two periods is 400% for asset market 1 and 750% for asset market 2. From $V_x$ and $V_y$, the Forbes and Rigobon adjusted correlation coefficient is $\nu_y = 0.5$. The distribution of asset returns for the precrisis and crisis periods under the null hypothesis are assumed to be bivariate normal with zero means and respective variance-covariance matrices given by $V_x$ and $V_y$. In computing the contagion tests, contagion is assumed to run from asset market 1 to asset market 2.

The finite sample distributions of the contagion statistics $FR$, $CS_1$ and $CS_2$, based on the Monte Carlo experiments are presented in Figure 3 using 100,000 replications. The precrisis sample size is $T_x = 147$ and the crisis sample size is $T_y = 109$. This choice is based on the sample size in the empirical application for the subprime crisis conducted below. For comparison, the Chi-squared asymptotic distribution with one degree of freedom, is also presented. The results show that the asymptotic distribution represents a good approximation of the finite sample distribution. Additional experiments using the sample sizes consistent with the Hong Kong crisis yield similar qualitative results.

4 Application to Real Estate Markets

The tests of contagion developed in Section 3 are now applied to identify potential contagious linkages stemming from real estate markets during the Hong Kong crisis of October 1997, and more recently from the US subprime mortgage crisis beginning July
2007. As real estate markets are important elements in both financial crises, it is pertinent to examine its role in more detail. Little empirical research has been conducted on contagion arising from real estate markets, with relevant references discussed in 4.1 and 4.2 below. See Appendix B for details of data sources. For both crises, three sets of contagious linkages are investigated:

1. Contagion across real estate markets of different countries.

2. Contagion from real estate to equity markets within countries.

3. Contagion from real estate to equity markets of different countries.

### 4.1 The Hong Kong Crisis

The Hong Kong crisis began on October 17, 1997, when returns on real estate in Hong Kong fell by 5.32% and equities by 4.29% in one day. Subsequently, returns in other real estate markets and equity markets in the region also fell (Quigley 2001; Kim and Lee 2002). Figure 4 gives time series of daily percentage returns for real estate and equity markets in Hong Kong, Japan, Singapore and the US from January 2, 1996, to June 30, 1998, with the precrisis period extending from January 2, 1996 to October 16, 1997 \((T_x = 469)\), and the crisis period from October 17, 1997 to June 30, 1998 \((T_y = 182)\). These dates are based on Bond, Dungey and Fry (2006).

The first set of linkages given above is investigated by Wilson and Zurbruegg (2004) and Bond et al. (2006). The second set of linkages are studied by Lu and So (2005) who test for Granger causality between real estate and equity markets. The third set of linkages are yet to be examined, although there is other work which focuses on cross-linkages amongst other asset classes (Granger, Huang and Yang 2000; Khalid and Kawai (2003); Dungey and Martin 2007). There are also a number of papers that study contagion during the Asian crisis between international equity markets (Baig and Goldfajn 1999; Forbes and Rigobon 2002; Caporale, Cipollini and Spagnolo 2003; Bekaert, Harvey and Ng 2005; Baur and Schulze 2005, amongst others). The majority of the empirical work is based on correlation tests of contagion which do not look at higher order moment interactions such as coskewness. The exceptions are Kallberg, Liu and Pasquariello (2002) who use Granger causality tests on both the levels and volatilities of asset returns, where the latter test can be viewed as a test of cokurtosis in
returns, and Knight, Lizieri and Satchell (2005) who also focus on higher order moments by testing the tails of the joint distribution using copulas (also see Bae, Karolyi and Stulz 2003, who test for contagion in the tails of the distribution using exceedances).

Table 2 gives the results of the Forbes and Rigobon (FR) and the coskewness ($CS_1$ and $CS_2$) contagion tests from the Hong Kong real estate market to selected regional markets. Given the Monte Carlo experiments presented above, the p-values reported in parentheses are based on the asymptotic distribution. In computing the conditional moments, an 8-variate VAR with five lags is estimated containing the real estate and equity returns of Hong Kong, Japan, Singapore and the US. The residuals from the VAR are used as the adjusted returns that are net of the market fundamentals, in the formulae for the Forbes and Rigobon contagion statistic in (38) and the coskewness contagion statistics in (44) and (45). Following the empirical contagion literature, the US is included in the VAR to represent a global common factor (Forbes and Rigobon 2002). To test for potential misspecification problems arising from omitted variables for example, an overall test of autocorrelation on the residuals based on the Lagrange mul-
Test of contagion from the Hong Kong securitized real estate market to selected asset markets: p-values in parentheses based on asymptotic critical values. All returns are adjusted for market fundamentals by estimating a VAR.

<table>
<thead>
<tr>
<th>Country</th>
<th>Real Estate</th>
<th>Equity</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$FR^{(a)}$</td>
<td>$CS_1^{(b)}$</td>
</tr>
<tr>
<td>Hong Kong</td>
<td>n.a.</td>
<td>n.a.</td>
</tr>
<tr>
<td></td>
<td>(0.00)</td>
<td>(0.00)</td>
</tr>
<tr>
<td>Japan</td>
<td>0.67</td>
<td>3.10</td>
</tr>
<tr>
<td></td>
<td>(0.41)</td>
<td>(0.08)</td>
</tr>
<tr>
<td>Singapore</td>
<td>0.17</td>
<td>96.55</td>
</tr>
<tr>
<td></td>
<td>(0.68)</td>
<td>(0.00)</td>
</tr>
</tbody>
</table>

(a) $FR$: Forbes and Rigobon contagion test.
(b) $CS_1$: Coskewness contagion test with coskewness measured in terms of Hong Kong real estate returns and squared returns of other markets.
(c) $CS_2$: Coskewness contagion test with coskewness measured in terms of squared Hong Kong real estate returns and returns of other markets.

tiplier test of autocorrelation yields a p-value of 0.696, showing that the null hypothesis of no autocorrelation cannot be rejected at the 5% level.

The results of the contagion tests in Table 2 show that the correlation based test of contagion given by $FR$, detects evidence of contagion from Hong Kong real estate to Hong Kong equities, but reveals no other evidence of contagion. In contrast, the coskewness tests of contagion given by $CS_1$ and $CS_2$, reveal that contagion is more pervasive across asset markets through the higher moments. The coskewness contagion tests identify evidence of contagion through either $CS_1$ or $CS_2$, or both, in all of the other markets investigated. In the case of the Japanese asset markets, the significant contagion channels are from volatility in Hong Kong real estate returns to the level of returns in real estate and equity markets in Japan. Perhaps the most striking differences in the correlation and coskewness contagion tests are for the real estate and equity markets in Singapore where the correlation test finds no evidence of contagion whereas the coskewness contagion test reveals highly significant evidence of contagion.
stemming from both the level and volatility in Hong Kong real estate returns. In terms of the model given by (14), the results suggest that it is the quantity risk terms

\[ E \left[ (r_{1,t+1} - \mu_{1,t+1})^2 (r_{2,t+1} - \mu_{2,t+1}) \right| s_{t+1} \right], E \left[ (r_{2,t+1} - \mu_{2,t+1})^2 (r_{1,t+1} - \mu_{1,t+1}) \right| s_{t+1} \right],

which are contributing to a change in the trade-off between the mean and the standard deviation of returns in the asset markets of Singapore (country 2) during the real estate crisis in Hong Kong (country 1).

4.2 US Subprime Crisis

The US subprime crisis began in mid 2007 with daily volatility in US real estate returns increasing from 1.57% to 3.89%, with the largest fall of −5.66% occurring on December 5, 2007. Comparable increases in volatility in real estate and equity returns for a diverse range of countries ensued as is evident in Figure 5 which presents daily percentage returns in 2007 beginning January 2 and ending December 25, for Australia, Germany, Hong Kong, Japan, the UK and the US. The precrisis period extends from January 2, 2007 to July 25, 2007 (\( T_x = 147 \)), while the crisis period is from July 26, 2007 to December 25, 2007 (\( T_x = 109 \)). Given its recency, little empirical work has been undertaken on the subprime crisis. Exceptions are Dungey, Fry, González-Hermosillo, Martin and Tang (2007) who focus on the second moments of the crisis by examining a factor structure, and Reinhart and Rogoff (2008) who compare the subprime crisis with historical crises. The starting date of the subprime crisis is based on Dungey et al. (2007).

Table 3 gives the results of the Forbes and Rigobon (\( FR \)) and the coskewness (\( CS_1 \) and \( CS_2 \)) contagion tests from the US real estate market to the global markets selected above, with p-values reported in parentheses based on the asymptotic distribution. In computing the conditional moments, a 12-variate VAR with one lag containing the real estate and equity returns of the 6 countries is estimated. The residuals from the VAR are used as the adjusted returns in the formulae for the Forbes and Rigobon contagion statistic in (38) and the coskewness contagion statistics in (44) and (45). The Lagrange multiplier autocorrelation test on the residuals yields a p-value of 0.108, in which case the null hypothesis of no autocorrelation cannot be rejected at the 5% level.

The results of the contagion tests in Table 3, show that the \( FR \) test provides evidence of contagion from US securitized real estate returns to the equity markets
in the UK and the US. There is no evidence of correlation based contagion in the equivalent real estate markets in the sample. As with the Hong Kong crisis application, the $CS_1$ and $CS_2$ coskewness tests of contagion find additional linkages not detected by the correlation based test for contagion. The key channels detected by the coskewness tests are from the level in the US real estate market to the volatility of Australian real estate and Hong Kong equity markets, and also from the volatility in the US real estate market to the level of the real estate and equity markets in Germany, as well as to the Hong Kong equity market. In contrast, the evidence for the UK is that it is its equity market that is affected by contagion from the US real estate market and that this channel is detected via a change in correlation and not through the higher order moments.

5 Application to Exchange Options

In this section the implications of changes in the comoments of the distribution of returns arising from contagion, are investigated in the context of pricing European exchange options. A European exchange option is the right to exchange one asset for another asset at time $T$, the time when the contract matures. Let $P_{i,t}$ be the price of asset $i = 1, 2$, at time $t$ with dividend yields $q_i$, $i = 1, 2$. The option price is the expected value of its discounted payoff (Hull 2000, p.470)

$$C_t = e^{-r_f(T-t)}E \left[ \max (P_{1,T} - P_{2,T}, 0) | s_{t+1} \right],$$

where the option contract is written in terms of exchanging asset 2 for asset 1, and $r_f$ is the risk-free rate of interest. The operator $E \left[ \cdot \right]$ is a conditional expectations operator based on information at time $t$ and the states of nature $s_{t+1}$.

In the case of Black-Scholes, asset prices are assumed to follow geometric Brownian motion. Under risk neutrality the stochastic differential equation of asset prices is specified as

$$dP_i = (q_i - r_f)P_idt + \sigma_i P_idW_i,$$

where $\sigma_i$ is the instantaneous volatility of the price of the $i^{th}$ asset, and $W_i$ is a Wiener process with the property that $dW_i \sim N(0, dt)$ and $E[dW_idW_j] = \rho dt$ allows for the Wiener processes to be correlated. Given (46) and (47) the Black-Scholes price of the
Figure 5: Daily returns (%) on real estate and equity markets for Australia (AU), Germany (GE), Hong Kong (HK), Japan (JP), the United Kingdom (UK) and the United States (US): January 2, 2007 to December 25, 2007.
Table 3:
Test of contagion from the US securitized real estate market to selected asset markets: p-values in parentheses based on asymptotic critical values. All returns are adjusted for market fundamentals by estimating a VAR.

<table>
<thead>
<tr>
<th>Country</th>
<th>Real Estate</th>
<th></th>
<th></th>
<th>Equity</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$FR_1^{(a)}$</td>
<td>$CS_1^{(b)}$</td>
<td>$CS_2^{(c)}$</td>
<td>$FR_1^{(a)}$</td>
<td>$CS_1^{(b)}$</td>
<td>$CS_2^{(c)}$</td>
</tr>
<tr>
<td>Australia</td>
<td>1.67</td>
<td>12.94</td>
<td>0.65</td>
<td>0.20</td>
<td>0.46</td>
<td>0.13</td>
</tr>
<tr>
<td></td>
<td>(0.20)</td>
<td>(0.00)</td>
<td>(0.42)</td>
<td>(0.66)</td>
<td>(0.50)</td>
<td>(0.72)</td>
</tr>
<tr>
<td>Germany</td>
<td>1.07</td>
<td>0.12</td>
<td>4.98</td>
<td>2.33</td>
<td>0.03</td>
<td>0.82</td>
</tr>
<tr>
<td></td>
<td>(0.30)</td>
<td>(0.73)</td>
<td>(0.03)</td>
<td>(0.13)</td>
<td>(0.86)</td>
<td>(0.36)</td>
</tr>
<tr>
<td>Hong Kong</td>
<td>0.27</td>
<td>0.54</td>
<td>7.5</td>
<td>0.29</td>
<td>0.65</td>
<td>5.55</td>
</tr>
<tr>
<td></td>
<td>(0.60)</td>
<td>(0.46)</td>
<td>(0.01)</td>
<td>(0.59)</td>
<td>(0.42)</td>
<td>(0.02)</td>
</tr>
<tr>
<td>Japan</td>
<td>0.50</td>
<td>2.29</td>
<td>0.55</td>
<td>0.69</td>
<td>2.10</td>
<td>0.4</td>
</tr>
<tr>
<td></td>
<td>(0.48)</td>
<td>(0.13)</td>
<td>(0.46)</td>
<td>(0.41)</td>
<td>(0.15)</td>
<td>(0.53)</td>
</tr>
<tr>
<td>UK</td>
<td>0.00</td>
<td>0.14</td>
<td>1.84</td>
<td>5.41</td>
<td>0.57</td>
<td>0.68</td>
</tr>
<tr>
<td></td>
<td>(0.95)</td>
<td>(0.71)</td>
<td>(0.18)</td>
<td>(0.02)</td>
<td>(0.45)</td>
<td>(0.41)</td>
</tr>
<tr>
<td>US</td>
<td>n.a.</td>
<td>n.a.</td>
<td>n.a.</td>
<td>15.12</td>
<td>4.96</td>
<td>2.29</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(0.00)</td>
<td>(0.03)</td>
<td>(0.13)</td>
</tr>
</tbody>
</table>

(a) $FR$: Forbes and Rigobon test of contagion.
(b) $CS_1$: Coskewness test of contagion with coskewness measured in terms of US real estate returns and squared returns of other markets.
(c) $CS_2$: Coskewness test of contagion with coskewness measured in terms of squared US real estate returns and returns of other markets.
exchange option is

\[ C_t^{BS} = P_{1,t} e^{-q_1 (T - t)} N(d_1) - P_{2,t} e^{-q_2 (T - t)} N(d_2), \tag{48} \]

where \( N(d_i), i = 1, 2 \) are cumulative normal functions evaluated at \( d_i \), as given by

\[ d_1 = \frac{\ln(P_{1,t}) - \ln(P_{2,t}) + (q_2 - q_1 + 0.5 \sigma_1^2)(T - t)}{\sigma \sqrt{T - t}}, \]

\[ d_2 = d_1 - \sigma \sqrt{T - t}, \]

and

\[ \sigma^2 = \sigma_1^2 + \sigma_2^2 - 2 \rho \sigma_1 \sigma_2. \]

Equation (47) shows that asset returns are normally distributed with mean \( q_i - rf - 0.5 \sigma_i^2 \), variance \( \sigma_i^2 \) and correlation \( \rho \). To investigate the effects of a financial crisis on the price of exchange options between two assets, the normality assumption underlying asset returns in (47) is now replaced by the following form of the generalized normal distribution in (15)

\[ f(r_1, r_2) = \exp \left[ \theta r_1^2 + \theta_2 r_2^2 + \theta_3 r_1 r_2 + \theta_4 r_1^3 + \theta_5 r_1^2 r_2 + \theta_6 r_1 r_2^2 + \theta_7 r_1^3 + \theta_8 r_1^4 + \theta_9 r_2^4 - \eta \right], \tag{49} \]

where \( \theta \) is the normalizing constant such that \( \int \int f(r_1, r_2) dr_1 dr_2 = 1. \)

To compute the option price in (47) assuming that returns are based on the generalized normal distribution in (49), a Monte Carlo approach is adopted. The conditional expectation in (46) is replaced by simulating (47) \( H = 10,000 \) times using a discrete time step of \( \Delta t = 0.1 \). The option price of the generalized normal distribution is computed as the discounted payoff of the sample mean of the \( H \) simulation runs

\[ C_t^{GEN} = e^{-rf(T - t)} \frac{1}{H} \sum_{i=1}^{H} \text{Max} \left( P_{1,i,T} - P_{2,i,t}, 0 \right), \tag{50} \]

where the superscript represents the \( i^{th} \) replication. To improve the accuracy of the simulated option price, a control variate is used whereby the difference between the analytical Black-Scholes price in (48) and the Monte Carlo price based on standardized conditional normality, is added to the generalized normal distribution option price in (49).

The results of the option experiments are given in Table 4 where the option prices in the precrisis period are equal to the Black-Scholes prices, while in the crisis period they are equal to the generalized normal prices. To determine the potential size of
mispricing during the crisis period from ignoring the nonnormalities in the returns, the Black-Scholes price is also computed in the crisis period using the implied volatility and correlation of the returns based on the generalized normal distribution. The option prices are evaluated using the following inputs

\[ q_1 = 0.04, q_2 = 0.05, rf = 0.1, T = 1, \sigma_1 = \sigma_2 = 0.2, \rho = 0.5, \]

with the price of asset 1 set at \( P_{1,t} = 80 \), while the following prices of asset 2 are chosen as

\[ P_{2,t} = \{60, 70, 80, 90, 100\} . \]

Given that returns during the financial crises of the empirical applications above tend to exhibit positive skewness and coskewness, the bivariate generalized normal distribution is parameterized according to two experiments. In the first experiment the parameters are set at

\[
\theta_1 = \theta_2 = -\frac{0.5}{1-\rho^2}, \theta_3 = \frac{\rho}{1-\rho^2}, \theta_4 = 0.0, \theta_5 = 0.0, \theta_6 = 0.7, \theta_7 = 0.0, \theta_8 = \theta_9 = -0.1, \tag{51}
\]

which yields positive coskewness of 0.270 compared to a value of zero coskewness in the precrisis case. Figure 6 gives a plot of (49) using this parameterization. This distribution is bimodal with a mode around the precrisis mode of zero, and an additional mode in the positive quadrant.

In the second experiment the parameters are

\[
\theta_1 = \theta_2 = -\frac{0.5}{1-\rho^2}, \theta_3 = \frac{\rho}{1-\rho^2}, \theta_4 = 0.0, \theta_5 = 0.5, \theta_6 = 0.0, \theta_7 = 0.0, \theta_8 = \theta_9 = -0.1, \tag{52}
\]

which yields positive coskewness of 0.354. Figure 7 gives a contour plot of (49) using this parameterization. In contrast to the distribution in Figure 6, this distribution is unimodal with part of the mass of the distribution extending out into the positive quadrant.

The precrisis distribution of returns is given by the standardized bivariate normal distribution with zero means, unit variances and correlation \( \rho \), which is obtained by imposing the additional restrictions \( \theta_i = 0, i > 3 \), on the parameters in either (51) or (52).

The results of the two experiments are given in Table 4 for alternative prices of Asset 2. In Experiment I, the effect of positive coskewness is to increase \( C_t^{GEN} \) in the crisis
Figure 6: Crisis bivariate distribution of returns in Experiment I.

Figure 7: Crisis bivariate distribution of returns in Experiment II.
Table 4:

Prices of European exchange options for precrisis and crisis periods, where the precrisis distribution is bivariate normal and the crisis distribution is bivariate generalized normal: $C_{t}^{BS}$ is the Black-Scholes price in the crisis period which incorrectly is based on bivariate normality and $C_{t}^{GEN}$ is the generalized normal price which correctly allows for positive coskewness in the crisis period.

<table>
<thead>
<tr>
<th>Price of Asset 2</th>
<th>Precrisis</th>
<th>Crisis (Experiment I)</th>
<th>Crisis (Experiment II)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$C_{t}^{BS}$</td>
<td>$C_{t}^{GEN}$</td>
</tr>
<tr>
<td>60</td>
<td>20.187</td>
<td>19.978</td>
<td>26.760</td>
</tr>
<tr>
<td>70</td>
<td>12.255</td>
<td>11.652</td>
<td>14.801</td>
</tr>
<tr>
<td>80</td>
<td>6.482</td>
<td>5.613</td>
<td>6.826</td>
</tr>
<tr>
<td>90</td>
<td>3.011</td>
<td>2.231</td>
<td>2.753</td>
</tr>
<tr>
<td>100</td>
<td>1.251</td>
<td>0.749</td>
<td>1.011</td>
</tr>
</tbody>
</table>

period above its precrisis value for the in-the-money options $P_{2,t} = \{60, 70\}$, and the at-the-money option $P_{2,t} = \{80\}$. For the out-of-the-money options $P_{2,t} = \{90, 100\}$, $C_{t}^{GEN}$ is below its precrisis prices. A comparison of the generalized exponential normal prices with the Black-Scholes prices in the crisis period reveals that Black-Scholes underprices options at just over 25% for all of the spot prices of asset 2 investigated. Similar results occur in Experiment II with the exception that for the extreme out-of-the-money option contract $P_{2,t} = \{100\}$, where Black-Scholes now over-prices the option compared to the generalized normal distribution price.

6 Conclusions

This paper introduced a new class of tests of contagion based on testing for changes in higher order comoments during financial crises, with special emphasis given to coskewness. The tests extended the existing correlation based class of contagion tests by identifying changes in contagion through the interaction of the level and volatility of returns. This was shown to be an important and natural linkage to test for contagion as it captured the portfolio effects of financial crises extending from higher order moments where the expected excess return on assets was expressed in terms of risk prices which were a function of the risk preferences of investors, and risk quantities which were a function of higher order conditional moments including skewness and coskewness. In deriving the tests of contagion a new family of bivariate distributions was presented.
This family was based on the generalized exponential distribution and extended the bivariate normal distribution to allow for higher order moments.

Two applications of the framework were presented: one empirical and the other theoretical. The empirical application consisted of applying the new tests to identifying contagious linkages stemming from real estate markets to other asset markets both nationally and globally during the Hong Kong crisis in October, 1997 and the US based subprime crisis in the second half of 2007. The higher-order moment contagion tests identified linkages across markets arising from contagion that were not detected in the correlation based measures of contagion, indicating the importance of accounting for higher order moments during crisis periods. The second application investigated the implications of changes in the returns distribution during financial crises for pricing options. The results showed that in the case of pricing exchange options, Black-Scholes can underprice options by 25% when returns exhibit positive coskewness during a financial crisis.

The contagion tests discussed and applied have been based on correlation and coskewness tests of contagion. However, the framework presented here suggests that other tests of contagion can also be adopted including tests based on cokurtosis for example. The analysis can also be extended to test for joint channels of contagion through a range of higher order moments. A characteristic of the tests presented here is that they use nonoverlapping samples as they involve comparing a precrisis sample with a crisis sample. An alternative strategy is to compare the crisis period with the total period (Forbes and Rigobon 2002). In deriving the standard errors of this statistic, the standard errors reported here would need to be augmented to include an additional source of dependence arising from using overlapping data.

Finally, the application to options which looked at the sensitivity of option prices to higher order comoments represents part of an existing research area that looks at relaxing the assumptions underlying Black-Scholes prices. In this application, the approach focused on exchange options written on two assets where the dependence of the returns on the two assets occurred at the second and third order moments and that this dependence changed during a financial crisis. It would be of interest to extend the analysis of exchange options to other exotic options which allow for contacts written on more than two assets, rainbow options, or where the contract is written on a portfolio
of assets, basket options.

A Derivation of Test Statistics

This Appendix gives the key derivations of the test statistics presented in the paper. Further details of these results, as well as derivations of a joint test of coskewness and dependence, and are available from the working paper version of the paper, Fry, Martin and Tang (2008).

A.1 Derivation of Coskewness Statistic

Consider the following generalization of the bivariate normal distribution

\[
\begin{align*}
    f(r_{1,t}, r_{2,t}) &= \exp \left[-\frac{1}{2} \left( \frac{1}{1 - \rho^2} \right) \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 + \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 
    - 2\rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) + \phi \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \right],
\end{align*}
\]

where

\[
\eta = \ln \iint \exp \left[-\frac{1}{2} \left( \frac{1}{1 - \rho^2} \right) \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 + \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 
    - 2\rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) + \phi \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \right] dr_1 dr_2
\]

\[
= \ln \iint \exp [h] dr_1 dr_2,
\]

and

\[
\begin{align*}
    h &= -\frac{1}{2} \left( \frac{1}{1 - \rho^2} \right) \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 + \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 
    - 2\rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) + \phi \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2.
\end{align*}
\]

A test of bivariate normality is based on the null hypothesis

\[ H_0 : \phi = 0. \]
Under the null hypothesis of bivariate normality the maximum likelihood estimators of the unknown parameters are simply
\[
\hat{\mu}_i = \frac{1}{T} \sum_t r_{i,t}; \quad \hat{\sigma}_i^2 = \frac{1}{T} \sum_t (r_{i,t} - \hat{\mu}_i)^2; \quad \hat{\rho} = \frac{1}{T} \sum_t \left( \frac{r_{1,t} - \hat{\mu}_1}{\hat{\sigma}_1} \right) \left( \frac{r_{2,t} - \hat{\mu}_2}{\hat{\sigma}_2} \right).
\] (57)

Let the parameters of (53) be
\[
\theta = \{ \mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho, \phi \}.
\]
The log-likelihood at time \( t \) is
\[
\ln L_t = -\frac{1}{2} \left( \frac{1}{1 - \rho^2} \right) \left[ \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 + \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \right] - 2\rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) + \phi \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 - \eta
\] = \( h_t - \eta. \) (58)

where
\[
h_t = -\frac{1}{2} \left( \frac{1}{1 - \rho^2} \right) \left[ \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right)^2 + \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \right] - 2\rho \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right) + \phi \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2,
\]
and \( \eta \) is given by (54).

Using equation (21) and the properties of the bivariate normal distribution, the information matrix under \( H_0 \) is
\[
I = -E \left[ \frac{\partial^2 \ln L}{\partial \Theta_i \partial \Theta_j} \right]_{\phi=0} = \begin{pmatrix}
\frac{1}{\sigma_1^2} & -\frac{\rho}{\sigma_1 \sigma_2} & 0 & 0 & 0 & \frac{(1 - \rho^2)}{\sigma_1} \\
-\frac{\rho}{\sigma_1 \sigma_2} & \frac{1}{\sigma_2^2} & 0 & 0 & 0 & \frac{2 \rho (1 - \rho^2)}{\sigma_2} \\
0 & 0 & \frac{2 - \rho^2}{4 \sigma_1^4} & -\frac{\rho^2}{4 \sigma_1^2 \sigma_2^2} & -\frac{\rho}{2 \sigma_1^2} & 0 \\
0 & 0 & -\frac{\rho^2}{4 \sigma_1^2 \sigma_2^2} & \frac{4 \sigma_2^4}{4 \sigma_2^4} & -2 \sigma_2^2 & 0 \\
0 & 0 & -\frac{\rho}{2 \sigma_2^2} & -\frac{\rho}{2 \sigma_2^2} & \frac{1 + \rho^2}{1 - \rho^2} & 0 \\
\frac{(1 - \rho^2)}{\sigma_1} & \frac{2 \rho (1 - \rho^2)}{\sigma_2} & 0 & 0 & 0 & \frac{(3 + 12 \rho^2)(1 - \rho^2)}{2 \sigma_2^2}
\end{pmatrix}.
\] (59)
Evaluating the gradient for $\phi$ under the null gives
\[
\frac{\partial \ln L}{\partial \phi} = \sum_{t=1}^{T} \left( \frac{\partial h}{\partial \phi} - T \left( \frac{\partial \eta}{\partial \phi} \right) \right)
= \sum_{t=1}^{T} \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 - T \left( E \left( \frac{\partial h}{\partial \phi} \right) \right)
= \sum_{t=1}^{T} \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 - T(0).
\] (60)

The gradient vector under $H_0$ is
\[
G = \begin{bmatrix} 0 & 0 & 0 & 0 & \sum_{t=1}^{T} \left( \frac{r_{1,t} - \mu_1}{\sigma_1} \right) \left( \frac{r_{2,t} - \mu_2}{\sigma_2} \right)^2 \end{bmatrix}.
\] (61)

The Lagrange multiplier statistic is obtained by substituting (59) and (61) into
\[
LM = G'I^{-1}G,
\] (62)
and replacing the unknown population parameters by consistent estimators under the null hypothesis. The pertinent statistic is
\[
LM = - \left( \frac{1 - \hat{\rho}^2}{-2\hat{\rho}^2 + 4\hat{\rho}^4 - 2} \right) \frac{1}{T} \sum_{t=1}^{T} \left( \frac{r_{1,t} - \hat{\mu}_1}{\hat{\sigma}_1} \right) \left( \frac{r_{2,t} - \hat{\mu}_2}{\hat{\sigma}_2} \right)^2
= \frac{T}{4\hat{\rho}^2 + 2} \left( \frac{1}{T} \sum_{t=1}^{T} \left( \frac{r_{1,t} - \hat{\mu}_1}{\hat{\sigma}_1} \right) \left( \frac{r_{2,t} - \hat{\mu}_2}{\hat{\sigma}_2} \right)^2 \right)^2.
\] (63)

which is asymptotically distributed as $\chi^2_1$ under the null hypothesis.

A.2 Derivation of the FR Statistic

In this Appendix the pertinent variance used in the FR test is derived. Consider
\[
Var \left( \hat{v}_{y|x_i} - \hat{\rho}_x \right) = Var \left( \hat{v}_{y|x_i} \right) + Var \left( \hat{\rho}_x \right) - 2Cov \left( \hat{v}_{y|x_i}, \hat{\rho}_x \right),
\]
where
\[
\hat{v}_{y|x_i} = \frac{\hat{\rho}_y}{\sqrt{1 + \hat{\delta} (1 - \hat{\rho}_y^2)}} \hat{\delta} = \left( \frac{s_{y,t}^2 - s_{x,t}^2}{s_{x,t}^2} \right).
\]
Define the empirical centered moments as

\[ m_{ij,k} = \frac{1}{k} \sum_{i} (r_{1,i} - \mu_{r_1})^i (r_{2,i} - \mu_{r_2})^j. \]

The first term is

\[
\text{Var} (\hat{\beta}_y | x_i) = \text{Var} \left( \frac{\hat{\beta}_y}{\sqrt{1 + \left( \frac{s^2_{y,i} - s^2_{x,i}}{s^2_{x,i}} \right) (1 - \hat{\rho}_y^2)}} \right)
\]

\[
= \text{Var} \left( \frac{m_{11,y}}{\sqrt{m_{20,y}m_{02,y}}} \right)
\]

\[
= \text{Var} \left( \frac{m_{20,y}m_{02,y}}{m_{11,y}m_{20,y}} \cdot \frac{1}{m_{02,y}} \cdot \frac{m_{02,y}}{m_{20,y}} + 1 \right)^{-\frac{1}{2}}
\]

The asymptotic standard errors are obtained from

\[
\text{Var} (\hat{\beta}_y | x_i) = \frac{1}{4} \left[ \frac{\mu_{20,y}}{\mu_{11,y}^2} \cdot \frac{1}{\mu_{20,y}^3} \cdot \frac{1}{\mu_{20,y}^2} \cdot \frac{1}{\mu_{20,y}^3} \cdot \frac{1}{\mu_{20,y}^3} \right] \text{Var} (m_{20,y})
\]

\[
+ \left[ \frac{\mu_{20,y}^2}{\mu_{11,y}^2} \right] \text{Var} (m_{02,y}) + \left[ \frac{-2\mu_{20,y}^2}{\mu_{11,y}^2} \right] \text{Var} (m_{11,y})
\]

\[
+ \left[ \frac{-2\mu_{20,y}^2}{\mu_{11,y}^2} \right] \text{Var} (m_{20,x}) + \left[ \frac{\mu_{20,y}^2}{\mu_{11,y}^2} \right] \text{Var} (m_{02,y}) + \left[ \frac{-2\mu_{20,y}^2}{\mu_{11,y}^2} \right] \text{Cov} (m_{20,y}, m_{02,y})
\]

\[
+ \left[ \frac{-2\mu_{20,y}^2}{\mu_{11,y}^2} \right] \text{Cov} (m_{20,y}, m_{11,y}) + \left[ \frac{\mu_{20,y}^2}{\mu_{11,y}^2} \right] \text{Cov} (m_{02,y}, m_{11,y})
\]

where \( \text{Var} (m_{20,y}) = \frac{1}{\mu_{y}} \left[ \mu_{40,y} - \mu_{20,y}^2 \right], \text{Var} (m_{02,y}) = \frac{1}{\mu_{y}} \left[ \mu_{04,y} - \mu_{02,y}^2 \right], \text{Var} (m_{11,y}) = \frac{1}{\mu_{y}} \left[ \mu_{22,y} - \mu_{11,y}^2 \right], \text{Var} (m_{20,x}) = \frac{1}{\mu_{x}} \left[ \mu_{40,x} - \mu_{20,x}^2 \right], \text{Cov} (m_{20,y}, m_{02,y}) = \frac{1}{\mu_{y}} \left[ \mu_{22,y} - \mu_{20,y}^2 \right]. \)
Cov \((m_{20,y}, m_{02,y}) = \frac{1}{T_y} \left[ \mu_{22,y} - \mu_{20,y} \mu_{02,y} \right] ; \) Cov \((m_{20,y}, m_{11,y}) = \frac{1}{T_y} \left[ \mu_{31,y} - \mu_{20,y} \mu_{11,y} \right] ;

Cov \((m_{02,y}, m_{11,y}) = \frac{1}{T_y} \left[ \mu_{13,y} - \mu_{02,y} \mu_{11,y} \right] ; \) and \( \mu_{20,y} = \sigma_{1,y}^2, \mu_{02,y} = \sigma_{2,y}^2, \mu_{31,y} = 3 \rho_y \sigma_{1,y}^2 \sigma_{2,y}^2, \mu_{13,y} = 3 \rho_y \sigma_{1,y}^2 \sigma_{3,y}^2, \mu_{40,y} = 3 \sigma_{1,y}^4, \mu_{04,y} = 3 \sigma_{2,y}^4, \mu_{11,y} = \rho_y \sigma_{1,y}^2 \sigma_{2,y}^2, \mu_{22,y} = (1 + 2 \rho_y^2) \sigma_{1,y}^2 \sigma_{2,y}^2, \) together with \( \mu_{20,x} = \sigma_{1,x}^2, \mu_{02,x} = \sigma_{2,x}^2, \mu_{40,x} = 3 \sigma_{1,x}^4, \) and \( \delta \equiv \frac{(\mu_{20,y} - \mu_{20,x})}{\sigma_{1,y}^2} - 1 = \frac{(\mu_{20,y} - \mu_{20,x})}{\sigma_{1,x}^2} - 1. \) Substituting these expressions into \( \text{Var} \left( \tilde{\nu}_{y|x} \right) \) yields

\[
\text{Var} \left( \tilde{\nu}_{y|x} \right) = \frac{1}{2} \left[ \frac{(1 + \delta)^2}{1 + \delta (1 - \rho_y^2)} \left( \frac{(2 - \rho_y^2) (1 - \rho_y^2)^2}{T_y} + \rho_y^2 (1 - \rho_y^2)^2 \right) \right].
\]

The second term is \( \text{Var} \left( \tilde{\rho}_x \right) = \frac{1}{T_x} (1 - \rho_x^2)^2, \) which immediately follows from the properties of the null distribution being bivariate normal.

The third and last term is

\[
\text{Cov} \left( \tilde{\nu}_{y|x}, \tilde{\rho}_x \right) = \text{Cov} \left\{ \frac{\tilde{\rho}_y}{1 + \left( \frac{s_{y,i}^2 - s_{x,i}^2}{s_{x,i}^2} \right) (1 - \tilde{\rho}_y)} \right\}
\]

\[
= \text{Cov} \left\{ \frac{m_{11,y}}{\sqrt{m_{20,y} m_{02,y}}}, \frac{m_{11,x}}{\sqrt{m_{20,x} m_{02,x}}} \right\}
\]

\[
= \text{Cov} \left\{ \left( \frac{m_{20,y} m_{02,y}}{m_{11,y} m_{20,x}} - \frac{m_{20,y}}{m_{20,x}} + 1 \right)^{-\frac{1}{2}}, \left( \frac{m_{20,x} m_{02,x}}{m_{11,x} m_{20,x}} \right)^{\frac{1}{2}} \right\},
\]

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which becomes

\[
Cov (\hat{\nu}_y | x, \hat{\nu}_x) = \left[ \begin{array}{cc}
-\frac{1}{4} \sqrt{\left[ \frac{\mu_{20,y} \mu_{02,y}}{\mu_{11,y} \mu_{20,x}} - \frac{\mu_{20,y}}{\mu_{20,x}} \right] + 1} & \frac{1}{\mu_{20,x} \mu_{02,y}} \\
\times \left[ \frac{-\mu_{20,y} \mu_{02,y}}{\mu_{11,y} \mu_{20,x}} + \frac{\mu_{20,y}}{\mu_{20,x}} \right] \left[ \frac{-\mu_{11,x}}{\mu_{20,x} \mu_{02,y}} \right] Var (m_{20,x}) + \\
\frac{-\mu_{20,y} \mu_{02,y}}{\mu_{11,y} \mu_{20,x}} + \frac{\mu_{20,y}}{\mu_{20,x}} \left[ \frac{-\mu_{11,x}}{\mu_{20,x} \mu_{02,y}} \right] Cov (m_{20,x}, m_{02,x}) + \\
\frac{-\mu_{20,y} \mu_{02,y}}{\mu_{11,y} \mu_{20,x}} + \frac{\mu_{20,y}}{\mu_{20,x}} \left[ \frac{-\mu_{11,x}}{\mu_{20,x} \mu_{02,y}} \right] Cov (m_{20,x}, m_{11,x}) + \\
\frac{1}{2 T_x} \rho_x \rho_y \left( 1 - \rho_y^2 \right) \left( 1 - \rho_x^2 \right) (1 + \delta),
\end{array} \right]
\]

where \( Var (m_{20,x}) = \frac{1}{T_x} \left[ \mu_{40,x} - \mu_{20,x}^2 \right] \), \( Cov (m_{20,x}, m_{02,x}) = \frac{1}{T_x} \left[ \mu_{22,x} - \mu_{20,x} \mu_{02,x} \right] \), \( Cov (m_{20,x}, m_{11,x}) = \frac{1}{T_x} \left[ \mu_{31,x} - \mu_{20,x} \mu_{11,x} \right] \), and \( \mu_{31,y} = 3 \rho_x \sigma_{1,x}^2 \sigma_{2,x}^2 \), \( \mu_{11,x} = \rho_x \sigma_{1,x} \sigma_{2,x} \), \( \mu_{22,x} = (1 + 2 \rho_x^2) \sigma_{1,x}^2 \sigma_{2,x}^2 \), using the results of the properties of bivariate normality.

\section*{B Data Sources}

The data consist of daily percentage returns of equity and real estate markets in Hong Kong, Japan, Singapore and the United States for the Hong Kong crisis and for Australia, Germany, Hong Kong, the United Kingdom and the United States for the subprime crisis. All series are denominated in US dollars. The returns are computed as the difference of the natural logarithms of daily price indices, multiplied by 100.

\textbf{Real Estate Data:} The real estate market indices are from the European Public Real Estate Association (http://www.epra.com). The mnemonics are: RUAU (Australia), RUGR (Germany), RUHK (Hong Kong), RUJP (Japan), RUSI (Singapore), RUUK (United Kingdom), and RUUS (United States).

The real estate indices are based on the share prices of large companies deriving at least 60 percent of their income from property related activities listed on the relevant stock exchange (Bond et al. 2006). As real estate returns are based on indices of
listed companies, they do not include unsecuritized assets and hence do not necessarily capture the characteristics of all real estate markets (Glascock, Lu and So 2000). This is not regarded as being too restrictive for the present empirical application, especially for Hong Kong whose real estate market is dominated by a small number of conglomerates (Fung and Forrest 2002).

**Equity Market Data:** The equity market indices are from Bloomberg. The mnemonics are: AS30 (Australia), HDAX (Germany), HSI (Hong Kong), TPX (Japan), SESALL (Singapore), ASX (United Kingdom) and SPX (United States).

**References**


