Pricing Default Events: Surprise, Exogeneity and Contagion

C. GOURIEROUX, A. MONFORT, J.-P. RENNE

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When investors are averse to a given risk, a security whose payoffs are exposed to this risk are less valuable than those whose payoffs are not.

A defaultable bond exposes its holder to two risks:

- (a) the risk that future probabilities of default change,
- (b) the risk that the bond issuer effectively defaults.

In order to derive closed form expressions of the prices of credit derivatives, most reduced-form models of credit risk "price" risk (a) but not the default events themselves (risk (b)).

That is, they implicitly consider that investors are not averse to the default-event surprise (or that these surprises can be diversified away).
A few papers mention this approximation and try to take into account the surprise, e.g.:

- Jarrow, Yu (2001, JoF) ["Counterparty Risk and the Pricing of Defaultable Securities"]. For 2 debtors only.
- A series of paper by Bai, Collin-Dufresne, Goldstein, Helwege (2013), with a very specific modeling of default dependence.

In general:

- Default dependence is difficult to specify.
- Derivative prices have no closed-form expressions.
This paper solves this problem for credit derivatives (CDS and CDO) written on a pool of credits, which can be partitioned into $J$ “large” homogenous segments.

The model accommodates different forms of contagion:

- exposure to common factors (frailty);
- self-exciting defaults;
- contagion across sectors.

Based on U.S. bond data, an application illustrates that this feature provides an explanation for the so-called credit-spread puzzle.
Outline of the presentation

1. Introduction.
2. The standard reduced-form approach and its limitations.
4. Applications.
2. The Standard Approach and its Limitations
Notations

- A pool of $I$ entities $i = 1, \ldots, I$.
- Default indicators $d_{i,t}$:
  \[
  d_{i,t} = \begin{cases} 
  1 & \text{if entity } i \text{ is in default at date } t, \\
  0 & \text{otherwise.}
  \end{cases}
  \]
- $n_t$ the number of defaults occurring at date $t$.
- $N_t = \sum_{\tau=1}^{t} n_{\tau}$ the cumulated number of defaults.
Assumptions on the historical distribution

i) Homogenous portfolio
The default indicators $d_{i,t+1}$ are independent, identically distributed given $F_{t+1}, d_t$.

ii) Default dependence driven by $F$
This conditional distribution depends on factor $F_{t+1}$ only.

iii) $F$ is exogenous
The conditional distribution of $F_{t+1}$ given $(F_t, d_t)$ is equal to the conditional distribution of $F_{t+1}$ given $F_t$.

Remark: (i) and (ii) will be relaxed in our general model.
The standard pricing approach

- Assumption on the stochastic discount factor:

\[ \tilde{m}_{t,t+1} = \tilde{m}(F_{t+1}). \]

- Then the price of a payoff \( g(N_{t+h}) \) at date \( t \) is:

\[
\tilde{\Pi}(g, h) = E_t[\tilde{m}_{t,t+1} \cdots \tilde{m}_{t+h-1,t+h}g(N_{t+h})]
= E_t[\tilde{m}_{t,t+1} \cdots \tilde{m}_{t+h-1,t+h}\tilde{g}(F_{t+h})],
\]

where: \( \tilde{g}(F_{t+h}) = E[g(N_{t+h})|F_{t+h}] \).

- Therefore: \( \tilde{\Pi}(g, h) = \tilde{\Pi}(\tilde{g}, h) \).

⇒ It is equivalent to price \( g(N_{t+h}) \) or to price its prediction \( \tilde{g}(F_{t+h}) \).
Risk premia associated with default events

- What is the change in pricing formula, when \( m_{t,t+1} = m(F_{t+1}, n_{t+1}) \)?
- Let us consider the projected sdf:
  \[
  \tilde{m}_{t,t+1} = E[m(F_{t+1}, n_{t+1}) | F_{t+1}].
  \]
- Then:
  \[
  \Pi(g, h) = \Pi(g, h) + \Pi(g - \tilde{g}, h).
  \]

**standard the price of formula surprise**
Relaxing the exogeneity assumption

- New assumption: The conditional historical distribution of $F_{t+1}$ given $F_t, d_t$ is equal to the conditional distribution of $F_{t+1}$ given $F_t, n_t$.

- A more complete decomposition of the derivative price:

$$\Pi(g, h) = \tilde{\Pi}(g, h) + [\Pi(\tilde{g}, h) - \tilde{\Pi}(\tilde{g}, h)] + \Pi(g - \tilde{g}, h)$$

price = standard + causality + surprise

adjustment adjustment
Moreover, we show that the standard formula for pricing a corporate zero-coupon bond:

\[ B(t, h) = \left( E_t^Q [\exp(-r_t \ldots - r_{t+h-1}) 1_{d_t, t+h=0}] \right) \]

\[ = E_t^Q [\exp(-r_t \ldots - r_{t+h-1} - \lambda_{t+1}^Q \ldots - \lambda_{t+h}^Q)], \]

cannot be used in general.

**Default intensities**

- If \( \Omega_t^* = (F_{t+1}, d_t) \), the **historical intensity** \( \lambda_{t+1} \) is defined by:

\[ P(d_{t+1} = 0 | d_t = 0, \Omega_t^*) = \exp(-\lambda_{t+1}). \]

- The **risk-neutral intensity** \( \lambda_{t+1}^Q \) is defined by:

\[ Q(d_{t+1} = 0 | d_t = 0, \Omega_t^*) = \exp(-\lambda_{t+1}^Q), \]

- If \( m_{t,t+1} = \exp(\delta_0 + \delta'_F F_{t+1} + \delta_S n_{t+1}) \), the **risk-neutral intensity** is:

\[ \lambda_{t+1}^Q = \lambda_{t+1} + \log\left\{ \exp(-\lambda_{t+1}) + [1 - \exp(-\lambda_{t+1})] \exp(\delta_S) \right\}. \]
3. Modeling Framework and Derivative Pricing
To get (quasi) closed form expressions for derivative prices, we need affine processes.

The joint process \((d_{1t}, \ldots, d_{lt}, F_t)\) cannot be affine, but the aggregate process \((n_t, F_t)\) can be if the size of the homogenous pool is large.
### Assumptions

(a) A Poisson regression model for the default count:

\[ n_{t+1} | F_{t+1}, n_t \sim \mathcal{P}(\beta' F_{t+1} + \beta n_t + \gamma); \]

(b) The conditional Laplace transform of \( F_{t+1} \) given \( F_t \) is exponential affine in \( (F_t, n_t) \):

\[ E_t[\exp(v' F_{t+1})] = \exp[a_F(v)' F_t + a_n(v)' n_t + b(v)]; \]

(c) The s.d.f. is exponential affine in both \( F_{t+1} \) and \( n_{t+1} \):

\[ m_{t,t+1} = \exp(\delta_0 + \delta_F F_{t+1} + \delta_S n_{t+1}). \]
In that setup, \((F_t, n_t)\) is jointly affine.

Then the price at date \(t\) of any exponential payoff \(\exp(uN_{t+h})\) can be derived by recursion.

Since the pool is homogenous, we know also how to price:
- individual default \(d_1\) (single name CDS),
- joint defaults \(d_1d_2\) ...

Indeed:

\[
\Pi(d_1 \ldots d_K, h) = \frac{1}{l(l-1)\ldots(l-K+1)} \left[ \frac{d^K}{d\nu^K} \Pi(\exp(N \log \nu), h) \right]_{\nu=1}.
\]

The price of a non-exponential payoff deduced by Fourier transform [Duffie, Pan, Singleton (2000)]: CDO pricing, tranches.
The pool can be partitioned into $J$ homogenous pools, with different risk characteristics.

For corporations, the segment can be defined by the industrial sector, by the size, by the domestic country, but the rating cannot be used since it is time-varying.

$n_{j,t}, j = 1, \ldots, J$ denote the numbers of defaults in each segment, conditionally independent:

$$n_{j,t+1} \sim \mathcal{P}[\beta_j F_{t+1} + C_j n_t + \gamma_j], j = 1, \ldots, J.$$ 

⇒ Additional contagion channel: across sectors.
4. Illustrations
App.1: Contagion and network

An illustration with six homogenous segments of size 100.

- Two types of factors:
  
  \[ F_{B,t} \] a sequence of i.i.d. Bernoulli variables
  
  \[ F_{N,t} \] processes keeping memory of past default counts in each segment
  
  \[ F_{N,j,t} = \rho F_{N,j,t-1} + n_{j,t-1}, j = 1, \ldots, 6. \]

- The distribution of the count variables with a circular structure of the network:
  
  \[ n_{1,t+1} \sim \mathcal{P}(0.4F_{N,6,t} + F_{B,t}), n_{j,t+1} \sim \mathcal{P}(0.4F_{N,j-1,t}), j = 2, \ldots, 6. \]
The next figure gives the evolutions of factors and default counts. A high value of factor $F_B$ may immediately generate defaults in segment 1. These defaults propagate to the other segments by contagion.
App.1: Contagion and network
App.1: Contagion and network

The next figure displays the term structures of:

- the CDS premium,
- the CDS without pricing the surprise,
- the actuarial value (physical probability)

for two dates and segments.
App.1: Contagion and network

Date 1

More than 5 defaults
between 1 and 5 defaults
No default

Date 45
Credit-spread puzzle: observation of a wide gap between

(a) Credit Default Swap (CDS) spreads, that can be seen as
default-loss expectations under the risk-neutral measure, and

(b) expected default losses (under $P$).
⇒ See e.g. D’Amato, Remolona (2003), Hull, Predescu, White (2005).

Standard credit-risk models, that do not price default-event
surprises, deal with the credit-risk puzzle by incorporating
credit-risk premia. But these premia are too small for short
maturities.

We show that pricing default-event surprises may solve the
credit-puzzle for all maturities, including the shortest ones.
We calibrate our model on U.S. banking-sector bond data covering the last two decades.

Specifically, we consider riskfree (Treasury) bonds and bonds issued by U.S. banks (1995-2013), rated BBB.

Our results suggest that neglecting the pricing of default events is likely to result in an overestimation of model-implied physical probabilities of defaults for short-term horizons.
App.2: Credit Spread Puzzle

<table>
<thead>
<tr>
<th></th>
<th>$\delta_{F,1}$</th>
<th>$\delta_{F,2}$</th>
<th>$\delta_{F,3}$</th>
<th>$\delta_{F,4}$</th>
<th>$\delta_S$</th>
<th>$\delta_0$</th>
</tr>
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<tbody>
<tr>
<td>M1</td>
<td>1</td>
<td>-0.974</td>
<td>3.045</td>
<td>-5.063</td>
<td>1.163</td>
<td>-0.044</td>
</tr>
<tr>
<td>M2</td>
<td>1</td>
<td>-0.972</td>
<td>5.681</td>
<td>-5.589</td>
<td>-</td>
<td>-0.081</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$\mu_1$</th>
<th>$\nu_1$</th>
<th>$\rho_1$</th>
<th>$\mu_2$</th>
<th>$\nu_2$</th>
<th>$\rho_2$</th>
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<tbody>
<tr>
<td>M1</td>
<td>1.55</td>
<td>0.022</td>
<td>0.95</td>
<td>0.428</td>
<td>0.004</td>
<td>0.95</td>
</tr>
<tr>
<td>M2</td>
<td>3.26</td>
<td>0.021</td>
<td>0.95</td>
<td>0.267</td>
<td>0.006</td>
<td>0.95</td>
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</table>

- M1 (resp. M2) is the model pricing the default-event surprise, i.e. with $\delta_S \neq 0$ (resp. $\delta_S = 0$).
- $F_{1,t}$ and $F_{2,t}$ follow independent ARG processes $[(\mu_1, \rho_1, \nu_1)$ and $(\mu_2, \rho_2, \nu_2)$, respectively].
- The sdf is given by $m_{t,t+1} = \exp(\delta_0 + \delta'_F F_{t+1} + \delta_S n_{t+1})$ where $F_t = [F_{1,t}, F_{1,t-1}, F_{2,t}, F_{2,t-1}]'$.
- The conditional distribution of $n_t$ given $F_t, n_{t-1}$ is Poisson $\mathcal{P}(F_{2,t})$.
- Calibration is carried out to reproduce a set of unconditional moments derived from observed data (fitted moments on next slide).
### Panel A - Unconditional moments (means / standard deviations)

| S: Sample, M1: model pricing the surprise, M2: model not pricing the surprise. |
|---|---|---|---|---|---|---|---|---|---|---|---|
|  | 1 mth | 1 y | 3 y | 5 y | 1 y | 3 y | 5 y | 1 y | 3 y | 5 y |
| ω | 50 | 50 | 50 | 50 | 100 | 100 | 100 | 0.05 | 0.05 | 0.05 |
| S | 2.7/2.1 | 3.1/2.2 | 3.5/2.0 | 3.9/1.8 | 2.0/1.6 | 2.5/1.8 | 2.8/2.0 | -60 | -70 | -65 |
| M1 | 2.7/2.2 | 3.1/2.1 | 3.6/1.9 | 3.9/1.8 | 1.7/1.9 | 2.3/1.8 | 3.1/1.8 | -65 | -65 | -65 |
| M2 | 2.6/1.7 | 3.1/1.7 | 3.8/1.8 | 3.8/2.3 | 0.6/0.9 | 1.3/1.3 | 2.6/2.4 | -47 | -54 | -70 |

### Panel B - Time-series fit (MSE divided by series variances, in %)

<table>
<thead>
<tr>
<th></th>
<th>1 mth</th>
<th>1 y</th>
<th>3 y</th>
<th>5 y</th>
<th>1 y</th>
<th>3 y</th>
<th>5 y</th>
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<td>2.8</td>
<td>0.2</td>
<td>3.1</td>
<td>11.1</td>
<td>1.2</td>
<td>7.2</td>
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<tr>
<td>M2</td>
<td>16.3</td>
<td>9.2</td>
<td>1.0</td>
<td>36.3</td>
<td>57.8</td>
<td>19.5</td>
<td>24.2</td>
</tr>
</tbody>
</table>

- M1 and M2 are estimated by weighted-moment methods (weights provided in row ω).
- Model M1 is better than M2 at reproducing sample moments, especially at the short-end of the term structure of spreads.
- Panel B reports the ratios of mean squared pricing errors (MSE) to the sample variances of corresponding yields/spreads.
- Pricing errors obtained with M1 are far lower than those associated with M2.
App.2: Credit Spread Puzzle

T-bond yields-to-maturity

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Data</th>
<th>Model M1 (which prices the surprise)</th>
<th>Model M2 (which does not price the surprise)</th>
<th>Model M1 under P</th>
<th>Model M2 under P</th>
</tr>
</thead>
<tbody>
<tr>
<td>1mth</td>
<td>2.5%</td>
<td>2.5%</td>
<td>2.5%</td>
<td>2.5%</td>
<td>2.5%</td>
</tr>
<tr>
<td>1y</td>
<td>3%</td>
<td>3%</td>
<td>3%</td>
<td>3%</td>
<td>3%</td>
</tr>
<tr>
<td>2y</td>
<td>3.25%</td>
<td>3.25%</td>
<td>3.25%</td>
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<tr>
<td>3y</td>
<td>3.5%</td>
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<tr>
<td>4y</td>
<td>3.75%</td>
<td>3.75%</td>
<td>3.75%</td>
<td>3.75%</td>
<td>3.75%</td>
</tr>
<tr>
<td>5y</td>
<td>4%</td>
<td>4%</td>
<td>4%</td>
<td>4%</td>
<td>4%</td>
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</tbody>
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Spreads

<table>
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<th>Maturity</th>
<th>Data</th>
<th>Model M1 (which prices the surprise)</th>
<th>Model M2 (which does not price the surprise)</th>
<th>Model M1 under P</th>
<th>Model M2 under P</th>
</tr>
</thead>
<tbody>
<tr>
<td>1mth</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
<td>0%</td>
</tr>
<tr>
<td>1y</td>
<td>1%</td>
<td>1%</td>
<td>1%</td>
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<td>1%</td>
</tr>
<tr>
<td>2y</td>
<td>1.5%</td>
<td>1.5%</td>
<td>1.5%</td>
<td>1.5%</td>
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<tr>
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<tr>
<td>4y</td>
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</table>
App.2: Credit Spread Puzzle
6. Conclusion
Standard approaches of credit-risk pricing neglect default-event surprises.

This paper proposes a tractable way to price these surprises.

In our framework, quasi-closed-form expressions for derivative prices still exist if the sizes of the homogenous segments are sufficiently large.

The specification accommodates different forms of contagion.

An empirical analysis suggests that models pricing default-event surprises can generate sizable credit-risk premia at the short end of the yield curve and, hence, can solve the credit-risk puzzle.
Appendix
If $F_t$ is exogenous under $P$ and $\delta_S \neq 0$, $F_t$ is no longer exogenous under $Q$.

The intensity $\lambda_{i,t}$ is a pre-intensity if:

$$P(\tau_i > t + h | \tau_i > t, \Omega^*_t) = E \left( \prod_{k=1}^{h} \exp(-\lambda_{i,t+k}|d_{i,t} = 0, \Omega^*_t) \right)$$

with $\tau_i = \inf\{t : d_{i,t} = 1\}$.

If $F_t$ is exogenous (under $P$), then $\lambda_{i,t+1}$ is a pre-intensity.

If $\delta_S \neq 0$, $F_t$ is not exogenous under $Q$ (even if it is exogenous under $P$) then $\lambda_{i,t+1}^Q$ is not a pre-intensity, and the standard formula for $B(t, h)$ is not valid.
In fact, the pricing formula becomes:

\[ B(t, h) = E_t \left[ \exp(-r_t \ldots - r_{t+h-1} - \tilde{\lambda}_Q^{t+1, t+h} \ldots - \tilde{\lambda}_Q^{t+h, t+h}) \right], \]

where

\[ \tilde{\lambda}_Q^{t+1, t+h} = -\log Q(d_{t+1} = 0 | d_t = 0, F_{t+h}) \]

is doubly indexed, with the interpretation of a "forward" intensity.
Homogenous model

- Factor: $F_t = (F_{1,t}, F_{1,t-1}, F_{2,t})$, where $(F_{1,t})$ and $(F_{2,t})$ are independent Autoregressive Gamma (ARG) processes.

A lagged value of $F_1$ is introduced to get more flexible specifications of the s.d.f. and of the term structure of the yields.

- Parameter $\beta$ is set in order to get: $n_t|F_t \sim P(F_{2,t})$

The short term rate is:

$$r_t = K_0 + K'_F F_t,$$

where the coefficients $K_0, K_F$ depend on the parameters characterizing the ARG dynamics, on the $\beta$, and on the parameter in the s.d.f. to ensure the AAO.
The next figure provides the evolutions of:

- the factors $F_1, F_2$,
- the short-term rate,
- the defaultable bond rate for the maturity $h = 20$,
- the spread between the latter and its "riskfree" counterpart (same maturity).
The next figure compares:
- the (forward) CDS price,
- this price without pricing the surprise,
- the cumulated probability of default.

(Forward) CDS prices to avoid the discounting effects that are implicit in the standard CDS pricing formula.

Note that the forward CDS prices are not exactly equal to the risk-neutral probability of default.

⇒ About half of the total credit-risk premia are accounted for by the credit-event risk premia.
⇒ This proportion weakly depends on the time-to-maturity.
Pricing Default Events: Surprise, Exogeneity and Contagion

Conclusion

(forward) CDS price
Without pricing of the surprise
Cumulated proba. of default

maturity

2% 4% 6% 8% 10% 12% 14% 16% 18% 20%

14%
12%
10%
8%
6%
4%
2%

2 4 6 8 10 12 14 16 18 20