WORKING paper

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A Unified Approach to Determinacy Conditions with Regime Switching

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Abstract

The conditions that ensure the existence of a unique stable equilibrium - determinacy conditions - for rational expectations models with Markov switching depend on the stability concept, contrasting with standard linear rational expectations models. In this paper, we offer a unified framework for the two commonly used stability concepts: boundedness and mean-square stability. We derive determinacy conditions for both concepts based on simple metrics. Qualitatively, we show that mean-square stable solutions are always at least as many as bounded solutions. We then apply and discuss our results in two monetary models.

Keywords: Markov-Switching, Indeterminacy, Monetary Policy.

JEL classification: E31, E43, E52

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NON-TECHNICAL SUMMARY

Standard dynamic stochastic general equilibrium models have recently been augmented to allow for regime-switching behavior of private agents and public authorities. This new class of models - known as Markov-switching rational expectations (MSRE) models - are proven to be powerful in explaining non-linear or time-varying macroeconomic phenomena. However, the underlying theory lacks unity and the literature has not yet reached a consensus on the determinacy conditions of such systems - conditions under which there exists a unique stable equilibrium.

There are various perspectives on how to interpret rational expectations models through the number of stable solutions. First, one can consider determinacy as a selection criterion among multiple models, interpreting the absence of stable solution or its multiplicity as a sign of misspecification. Second and quite contrary to the first interpretation, one can consider the number of stable equilibria as informative about true economic problems: for instance, the multiplicity of stable equilibria is sometimes interpreted as a consequence of bad monetary policy (see for instance Lubik and Schorfheide (2004) and the discussion about the Taylor principle). Regardless of the interpretation, identifying the number of stable solutions is a prerequisite for understanding a model and its ability to account for economic fluctuations.

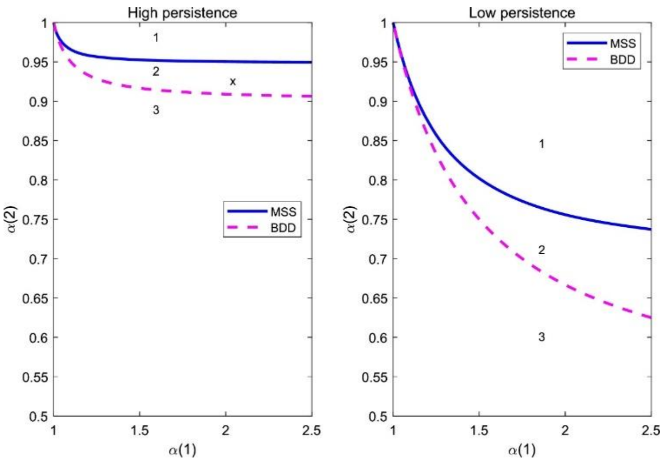
Since the earlier contributions of Davig and Leeper (2007) and Farmer et al. (2009b), identifying determinacy has been a subtle and difficult task. Due to the inherent non-linearity of regime-switching models, it is not possible to apply the well-known methodologies for linear models to this class of rational expectations models. So far, two major stability concepts to characterize determinacy have been used: boundedness and mean-square stability. Barthelemy and Marx (2019) and Cho (2021) have recently derived determinacy conditions for these two concepts. Not surprisingly, the determinacy conditions do not coincide with each other. But more surprisingly the methods used to derive the determinacy conditions seem very different raising the question of whether there exists a way to unify the two approaches. Such a lack of methodological consensus might be responsible for not using Markov-switching models in macroeconomics more frequently.

In this paper, we develop a unified framework that works for both stability concepts and allows to better understand the differences between the two stability concepts and their implications. Specifically, we consider a class of MSRE models for which determinacy using boundedness and mean-square stability concepts are well-defined. Our main contribution is to establish a complete classification result for MSRE models under the standard hypotheses assumed for mean-square stability. We derive necessary and sufficient conditions for determinacy, indeterminacy and the case of no stable solution under both stability concepts. Then, we show that the MSRE models have in total six partitions with always more mean-square-stable solutions than bounded ones. We also propose efficient ways to check determinacy in practice for the two stability concepts. Finally, we apply the results to two applications: a monetary model in the vein of Davig and Leeper (2008) and a model of monetary and fiscal interaction in the vein of Cho (2021).

As an illustration, we present below some results for the monetary model *à la* Davig and Leeper (2008). In this model, monetary policy follows a Taylor-like rule $i_t = \alpha(s_t)\pi_t$ where π_t is inflation; i_t stands for the nominal interest rate and the coefficient α measures the reactiveness of monetary policy to inflation. This coefficient can take two values $\alpha(1)$ and $\alpha(2)$ depending on the policy regime denoted by s_t, with transition probabilities p_{ij}. In addition, inflation is determined through a Fisherian equation linking the nominal interest rate, inflation, and the real interest rate supposed exogenous: $i_t = E_t\pi_{t+1} + r_t$. Plugging the switching Taylor rule into this equation shows that the higher α, the stronger the response of monetary policy to realized inflation and hence the lower the impact of future expectations on current inflation. This model is simpler than the general case we deal in the paper since there are no backward components in the equation. As a result, we get only three cases (instead of 6), which are represented in Figure 1. For values of $(\alpha(1), \alpha(2))$ above the blue line, there exists a unique MSS solution which is also bounded. For values of $(\alpha(1), \alpha(2))$ between the solid blue and dotted magenta lines, there exists a unique bounded equilibrium but several MSS equilibria. Below the magenta dotted line, there are several MSS and bounded equilibria. Three remarks are in order. First, the gap between the two determinacy frontiers crucially depends on the probability of switching from one regime to the other, for very persistent regimes (left figure), this gap is not quantitatively important. Second, the qualitative results do not depend on the exact stability concept that is used: for instance, in this calibrated and simple model, the more monetary policy reacts to inflation in one regime the less monetary policy needs to react in the other regime to rule out equilibrium multiplicity (indeterminacy) and self-fulfilling prophecies. Third, monetary policy should be more active (higher level of α's, and the more so for more persistent policy) to rule out equilibrium multiplicity under MSS than boundedness. In the fiscal-monetary model presented in the paper, we show that the existence of stable solution requires a more passive fiscal policy for boundedness compared to MSS.

More generally, this study proposes a complete methodological foundation for studying the class of macroeconomic models known as Markov-switching rational expectations models. We find that only six configurations are possible and the two stability concepts lead to the same conclusion in three of them. In any case, there are always more mean-square stable solutions than bounded ones in this class of models. Finally, we believe that our proposed approach is tractable enough and easy to apply for more advanced macroeconomic topics that involve strong non-linearities -e.g., Covid crisis, policy regime shift, financial crisis and so on- impossible to analyse in the context of standard linear models.

Figure 1. Classification of regime-switching Fisherian model



Note: This figure depicts the region for determinacy and indeterminacy for the Fisherian model in boundedness (BDD) and mean-square stability (MSS), for two sets of values of probabilities $p_{11}=p_{22}=0.9$ (high persistence, left panel), and $p_{11}=p_{22}=0.5$ (low persistence, right panel). Solid (Dashed) thick line separates the regions of determinacy and indeterminacy in mean-quare stability (boundedness) for $p_{11}=p_{22}=0.9$. The numbers 1, 2 and 3 represent respectively partition 1: determinacy in both stabilities, partition 2: determinacy in boundedness and indeterminacy in mean-square stability, partition 3: indeterminacy in both stabilities. The point x corresponds to a set of parameters $\alpha(1)=2$ and $\alpha(2)=0.93$ corresponding to a continuum of mean-square stable solutions and a unique bounded solution.

Une approche unifiée des conditions de détermination dans les modèles à changement de régimes.

RÉSUMÉ

Les conditions qui garantissent l'existence d'un équilibre stable unique – conditions de détermination- pour des modèles à anticipations rationnelles avec des changements de régime markoviens dépendent du concept de stabilité, à la différence des modèles linéaires à anticipations rationnelles. Dans ce papier, nous proposons un cadre unifié pour les deux concepts de stabilité usuellement utilisés : l'espace des processus bornés et celui des processus à espérance et variance bornées (mean-square stable). Nous exprimons les conditions de détermination pour ces deux concepts à l'aide de métriques matricielles simples. Nous montrons que les solutions à espérance et variance bornées sont toujours plus nombreuses que les solutions bornées. Nous appliquons et discutons nos résultats pour deux modèles monétaires.

Mots-clés : changement de régimes markovien, détermination, politique monétaire

1 Introduction

Standard dynamic stochastic general equilibrium models have recently been augmented by regime-switching behaviors of private agents and public authorities. This new class of models – known as Markov-switching rational expectations (MSRE) models – are proven to be powerful in understanding non-linear or time-varying macroeconomic phenomena.[1] However, the underlying theory lacks unity and especially the determinacy conditions — conditions under which there exists a unique stable equilibrium — have not yet reached a consensus.

There are various perspectives on how to interpret rational expectations models through the number of stable solutions. First, one can consider determinacy as a selection criterion among multiple models, interpreting the absence of stable solution or its multiplicity as a sign of misspecification. Second and quite contrary to the first interpretation, one can consider the number of stable equilibria as informative about true economic problems: for instance, the multiplicity of stable equilibria is sometimes interpreted as a consequence of bad monetary policy (see for instance Lubik and Schorfheide (2004) and the discussion about the Taylor principle). Regardless of the interpretation, identifying the number of stable solutions is a prerequisite for understanding a model and its ability to account for economic fluctuations.

Since the earlier contributions of Davig and Leeper (2007) and Farmer et al. (2009b), identifying determinacy has been a subtle and difficult task.[2] Due to the inherent non-linearity of regime-switching models, it is not possible to apply the well-known methodologies for linear models to this class of rational expectations models. So far, two major stability concepts to characterize determinacy have been used: boundedness and mean-square stability. Barthélemy and Marx (2019) and Cho (2021) have recently derived determinacy conditions for these two concepts. Not surprisingly, the determinacy conditions do not coincide with each other. But more surprisingly the methods used to derive the determinacy conditions seem very different raising the question whether there exists a way to unify the two approaches. Such a lack of methodological consensus might be responsible for not using Markov-switching models in macroeconomics more frequently.

In this paper we develop a unified framework that works for both stability concepts and allows to better understand the differences between the two stability concepts and their implications. Specifically, we set out the common set of models and the solution space for

[1] See Bianchi (2013), Sims and Zha (2006), Davig and Leeper (2008), Svensson et al. (2008), Baele et al. (2015), Bianchi and Melosi (2017) and Ascari et al. (2020) among others.

[2] Other important contributions include, among others, Farmer et al. (2009a, 2011), Cho (2016), Foerster et al. (2016) and Neusser (2019).

which determinacy using boundedness and mean-square stability concepts are well-defined. Our main contribution is to establish a complete classification result for MSRE models under the standard hypotheses assumed for mean-square stability. We derive necessary and sufficient conditions for determinacy, indeterminacy and the case of no stable solution under both stability concepts. Then, we show that the MSRE models have in total six partitions[3] with always more mean-square-stable solutions than bounded ones. We also propose efficient ways to check determinacy in practice for the two stability concepts. Finally, we apply the results to two applications: a monetary model in the vein of Davig and Leeper (2008) and a model of monetary and fiscal interaction in the vein of Cho (2021). In both examples, monetary policy should be more active to rule out equilibrium multiplicity under MSS than boundedness, whereas, in the fiscal-monetary model, existence of stable solution requires a more passive fiscal policy for boundedness compared to MSS.

This study is organized as follows. Section 2 sets out the common class of models and the solution space, and formally defines boundedness and mean-square stability for rational expectations solutions. Section 3 derives our main classification result. Section 4 applies our results to two economic examples. Section 5 concludes.

2 Setting

Environment We consider the following class of linear models with rational expectations and regime switching:

$$x_t = \mathbb{E}_t[A(s_t)x_{t+1}] + B(s_t)x_{t-1} + C(s_t)\epsilon_t, \ \forall t \geq 0, \tag{1}$$

where time is discrete and indexed by $t \geq 0$, x_t is a $n \times 1$ vector of endogenous variables with x_{-1} being initially given. ϵ_t is a $m \times 1$ vector of structural exogenous shocks. s_t is an ergodic Markov process valued in $\{1, 2, ..., S\}$ and the transition probability is defined as $p_{ij} = \Pr(s_{t+1} = j|s_t = i)$ for $i, j = 1, ..., S$ such that $\sum_{j=1}^{S} p_{ij} = 1$ for all i. $\mathbb{E}_t$ denotes the expectations operator given information available at date t. $A(\cdot), B(\cdot)$ and $C(\cdot)$ are conformable coefficient matrices of appropriate dimensions.[4] The matrices $A(\cdot)$ may be singular but we assume that their rank does not depend on the regime. As it is standard in

[3]In three of them, both stability concepts coincide: a unique, multiple and no stable solution under both stability criteria. The remaining three partitions are the cases of 1) a unique bounded stable solution and multiple mean-square solutions, 2) no bounded solution and a unique mean-square stable solution and 3) no bounded solution and multiple mean-square stable solutions.

[4]Appendix A shows how to deal with models in which A depends on future regime.

the literature, we assume that the vector x_t only includes variables that are undetermined at date t. A rational expectations solution is a stochastic process $\{x_t\}_{t\in\mathbb{N}}$ that solves model (1). Any such a solution can be put into the following convenient form:

$$x_t = [\Omega(s_t)x_{t-1} + \Gamma(s_t)\epsilon_t] + w_t, \tag{2}$$

$$w_t = F(s_t)\mathbb{E}_t[w_{t+1}], \tag{3}$$

where regime-dependent matrices Ω, Γ and F satisfy the following system of equations:

$$B(s_t) - [\mathbb{1}_n - \mathbb{E}_t(A(s_t)\Omega(s_{t+1}))]\,\Omega(s_t) = 0_{n\times n}, \tag{4a}$$

$$C(s_t) - [\mathbb{1}_n - \mathbb{E}_t(A(s_t)\Omega(s_{t+1}))]\,\Gamma(s_t) = 0_{n\times m}, \tag{4b}$$

$$A(s_t) - [\mathbb{1}_n - \mathbb{E}_t(A(s_t)\Omega(s_{t+1}))]\,F(s_t) = 0_{n\times n}. \tag{4c}$$

A process (2) is called a fundamental or minimum state variable (MSV) solution when $w_t = 0_{n\times 1}$. There are finitely many solutions for $\Omega(s_t)$, but $\Gamma(s_t)$ and $F(s_t)$ are uniquely associated with each $\Omega(s_t)$. Henceforth, solving for MSV solutions is equivalent to solving for $\Omega(s_t)$.

When $w_t \neq 0_{n\times 1}$, it is referred to as a sunspot process and (2) as a sunspot solution. Notice that (3) is a purely forward-looking homogenous model for w_t. Farmer et al. (2009b) show that any process that solves (3) can be put in the following form:

$$w_t = \Lambda(s_{t-1}, s_t)w_{t-1} + G(s_t)\eta_t, \tag{5}$$

where η_t is any stochastic process such that $\mathbb{E}_t[\eta_{t+1}] = 0_{n\times 1}$.[5]

Stability concepts We now introduce two well-known concepts of stability: bounded stability and mean-square stability.

Definition 1. ***(Bounded Process)*** *An $n-$dimensional stochastic process $\{x_t\}_{t\in\mathbb{N}}$ is bounded if there exists $N \in (0,\infty)$, such that $||x_t|| < N$ for any history and any date $t \geq 0$.*

As it is common in the literature, we also discard exotic solutions that do not converge to a steady-state.[6]

[5]Using (5), the general solution (2) can be equivalently written as: $x_t = [\Omega(s_t) + \Lambda(s_{t-1}, s_t)]x_{t-1} - \Lambda(s_{t-1}, s_t)\Omega(s_{t-1})x_{t-2} + \Gamma(s_t)\epsilon_t - \Lambda(s_{t-1}, s_t)\Gamma(s_{t-1})\epsilon_{t-1} + G(s_t)\eta_t$. This will also be abusively called a sunspot solution even when η_t depends on structural shocks.

[6]Interested reader may refer to Appendix.B for the technical details

Definition 2. ***(Mean-square stable Process)*** *An $n-$dimensional stochastic process $\{x_t\}_{t\in\mathbb{N}}$ is mean-square stable (MSS) if, for any initial condition x_{-1}, there exist an $n \times 1$ vector μ and an $n \times n$ matrix Σ such that $\lim_{t\to+\infty} \mathbb{E}_0(x_t) = \mu$ and $\lim_{t\to+\infty} \mathbb{E}_0(x_t x_t') = \Sigma$.*

Clearly, the two stability concepts are distinct. Boundedness relates to the classic $\mathcal{C}^\infty$ norm in the infinite space of stochastic processes whereas mean-square stability relates to the convergence of the first two moments. As such, determinacy conditions using the two stability concepts have been identified based on diverse hypotheses such as different assumptions on shock processes, and unharmonized approaches in the literature.

The main purpose of this study is to propose a unified single measure to characterize determinacy in both stability concepts. To do so, we make the two following assumptions.

Assumption 1. *The structural shock ϵ_t is bounded and mean-square stable.*

Assumption 2. *The sunspot shock η_t is bounded, mean-square stable and independent of w_{-1} and s_t for all $t \geq 0$.*

Only boundedness (mean-square stability) would be required for the shock processes if one is interested in examining a unique bounded (mean-square stable) rational expectations solution. Therefore, Assumptions 1 and 2 are restrictive for each stability concept. However, both boundedness and existence of finite first and second moments are well-accepted requirements for the shock processes in macroeconomics. More crucially, these assumptions enable us to clarify the determinacy conditions under two stability concepts using a single measure. Henceforth, Assumptions 1 and 2 are assumed to hold in what follows.

Unified Measures for the Two Stability Concepts Before presenting the main results, let us introduce the scalar $\mu_n(M, P)$ which measures the asymptotic explosiveness of a stochastic product of matrix M given the transition probability matrix P and a norm. Formally,

$$\mu_n(M, P) = \lim_{k\to+\infty} \left(\mathbb{E}\|M(s_0)\cdots M(s_k)\|^n\right)^{1/nk}. \tag{6}$$

where $s_t \in \{1, ..., S\}$ for all $t \geq 0$ and for all $n \in [1, ..., \infty]$. The exact definition of expectations in Equation (6) is given in Appendix C. Our measure extends the standard spectral radius introduced in Protasov (1997) for uniformly distributed matrices (see, for instance, Jungers and Protasov (2010)) to an environment with transition probabilities p_{ij}. In words, $\mu_n(M)$ measures the overall growth of the product of random matrices, for a given norm where P is skipped in our measure for brevity. Notice that the underlying norm $\|.\|$ is unimportant given the equivalence of norms in finite dimensional spaces.

The main results of this study critically hinge on the following ranking condition:

Lemma 1. *For a given set of argument matrices $\{M(s)\}_{s\in\{1,,\ldots,S\}}$, a fixed norm and probability P, the following holds:*

$$\mu_1(M) \leq \mu_2(M) \leq \mu_\infty(M) \tag{7}$$

Proof. See Appendix C. □

For the class of solutions to model (1), Lemma 1 implies that a fundamental solution $x_t = \Omega(s_t)x_{t-1} + \Gamma(s_t)\epsilon_t$ is bounded if and only if $\mu_\infty(\Omega') < 1$ and is MSS if and only if $\mu_2(\Omega') < 1$ under Assumption 1. That is, if a fundamental solution is bounded, it is also mean-square stable. By the same token, if a sunspot process is bounded, it is mean-square stable under Assumption 2. Henceforth, there are always at least as many MSS solutions that bounded ones. However, determinacy in boundedness (MSS) is the case in which there exists a unique bounded (MSS) fundamental solution and *non-existence* of bounded (MSS) sunspot processes. This explains why determinacy in boundedness is neither necessary nor sufficient for that in mean-square stability in a general setup under regime-switching as we prove in the following section.

3 Determinacy conditions

For an expositional purpose, we first derive the determinacy conditions in forward-looking models. We then extend to models with backward-looking components.

Forward-looking models For models without backward looking components in which $B(s_t) = 0_{n\times n}$, there always exists a unique fundamental solution $x_t = C(s_t)\epsilon_t$, and it is both bounded and mean-square stable under Assumption 1. Therefore, determinacy is equivalent to the case of no stable sunspot processes subject to Equation (3) with $F(s_t) = A(s_t)$ (see Equation (4c)). Our innovation is to show that such a condition is simply characterized by our proposed unified measure, without solving for all of the sunspot processes, and examining their stability. Formally,

Proposition 1. *For Model (1) with $B(s_t) = 0_{n\times n}$, there exists a unique bounded (mean-square stable) solution if and only if $\mu_1(A) \leq 1$ ($\mu_2(A) \leq 1$). Moreover, $\mu_1(A) \leq \mu_2(A)$.*

Proof. See Appendix D. □

Proposition 1 highlights that a unique mean-square stable solution implies a unique bounded solution, but the converse is not true.[7] Moreover, Proposition 1 partitions the entire class of forward-looking regime-switching models into three: if $\mu_2(A) \leq 1$ there exists a unique solution under both stability concepts; if $\mu_1(A) \leq 1 < \mu_2(A)$ there is a unique bounded solution but multiple MSS ones; otherwise there are multiple MSS and bounded solutions. It is also important to remind that in the absence of regime-switching, $\mu_1(A) = \mu_2(A)$ and they are just the maximum absolute eigenvalue of A.

Generalization Assumptions 1 and 2 allow us to apply the strategy developed by Cho (2021) for mean-square stability to boundedness. This strategy is based on the analysis of the minimum of modulus (MOD) solution. Suppose that there are H fundamental solutions. Without loss of generality, these solutions are indexed such that $\mu_2(\Omega_1') \leq ... \leq \mu_2(\Omega_H')$. $\Omega_1(s_t)$ is referred to as the MOD solution in the mean-square stability sense. The same set of solutions can be similarly indexed for boundedness such that $\mu_\infty(\tilde{\Omega}_1') \leq ... \leq \mu_\infty(\tilde{\Omega}_H')$. $\tilde{\Omega}_1(s_t)$ is now the MOD solution in boundedness sense. Notice that $\Omega_1(s_t)$ can differ from $\tilde{\Omega}_1(s_t)$, even if this seems rare in practice.

As in Cho (2021), the characterization of the determinacy, indeterminacy and no stable solution regions can be done by analyzing the properties of the MOD solution. We can now state the main proposition that gives this characterization for both stability concepts.

Proposition 2. *Determinacy for the general MSRE model (1) is characterized as follows.*

1. *There exists a unique bounded solution if and only if* $\mu_1(\tilde{F}_1) \leq 1$ *and* $\mu_\infty(\tilde{\Omega}_1') < 1$

2. *There exists a unique MSS solution if and only if* $\mu_2(F_1) \leq 1$ *and* $\mu_2(\Omega_1') < 1$

Proof. See Appendix E. □

The Proposition's novelty is that the determinacy conditions for both stability concepts are perfectly symmetric thanks to Assumption 2. Assertion 1 restates Proposition 3 in Barthélemy and Marx (2019) using μ_1 and μ_∞ metrics and MOD solution but also gives

[7] The proof relies on the following observation. For mean-square stability, $\mu_2(A)\mu_2(\Lambda') \geq 1$ for all $\Lambda(s_t, s_{t+1})$ in Equation (5) subject to $w_t = A(s_t)\mathbb{E}[w_{t+1}]$ and there exists a $\Lambda^*(s_t, s_{t+1})$ such that $\mu_2(A)\mu_2(\Lambda^{*\prime}) = 1$. Hence, $\mu_2(A) \leq 1$ if and only if there is *no* mean-square stable sunspot process, i.e., $\mu_2(\Lambda') \geq 1$ for all $\Lambda(\cdot)$. For boundedness, the same logic holds with $\mu_1(A)$ and $\mu_\infty(\Lambda')$. Therefore, determinacy in mean-square stability ($\mu_2(A) \leq 1$) implies determinacy in boundedness ($\mu_1(A) \leq 1$).

a *necessary and sufficient* condition for a unique bounded solution.[8] Assertion 2 is the restatement of Proposition 3 in Cho (2021) for mean-square stability in terms of our unified measure.[9] It is important to note both two metrics jointly determines the existence and uniqueness of a bounded (MSS) solution, i,e., it is not the case that one metric is a condition for the existence and the other is for uniqueness. Now the following proposition presents the complete classification result for the MSRE models.

Proposition 3. *There are six mutually disjoint and exhaustive partitions for regime-switching rational expectations models under the two stability concepts given by Table 1:*

Table 1: Classification of the MSRE models by Boundedness and Mean-square Stability

$\mu_1(\tilde{F}_1) < \mu_2(F_1) \leq 1$	6. NSS(MSS) NSS(BDD)	4. DET(MSS) NSS(BDD)	1^*. DET(MSS) DET(BDD)
$\mu_1(\tilde{F}_1) \leq 1 < \mu_2(F_1)$		5. IND(MSS) NSS(BDD)	2. IND(MSS) DET(BDD)
$1 < \mu_1(\tilde{F}_1) < \mu_2(F_1)$			3. IND(MSS) IND(BDD)
	$1 \leq \mu_2(\Omega_1') < \mu_\infty(\tilde{\Omega}_1')$	$\mu_2(\Omega_1') < 1 \leq \mu_\infty(\tilde{\Omega}_1')$	$\mu_2(\Omega_1') < \mu_\infty(\tilde{\Omega}_1') < 1$

MSS and BDD in parenthesis stand for mean-square stability and boundedness. DET, IND and NSS represent the case of determinacy, indeterminacy and no stable solution. There are six different partitions. We assign numbers 1 through 6 to those six partitions. Empty entries indicate the cases in which it is not possible that the two conditions in the row and column hold. Partition 1^* is possible only if $\Omega_1 = \tilde{\Omega}_1$.

Proof. See Appendix F □

This proposition confirms that determinacy in boundedness is neither necessary nor sufficient for determinacy under MSS. A novel finding is that there are only six partitions of MSRE models. The two stability concepts lead to the same conclusion for partitions 1, 3 and 6 and differ in partitions 2, 4 and 5, which arise only for MSRE models: in the absence of regime-switching, partitions 2, 4 and 5 disappear, and $\tilde{\Omega}_1 = \Omega_1$ by construction.[10] So the important question is to understand how likely the partitions 2, 4 and 5 emerge in practice.

[8] Additionally, but less importantly, Assertion 1 also holds when $A(s_t)$ is singular.

[9] In his Proposition 3, Cho (2021) uses the two measures $\rho(\Psi_{F\otimes F})$ and $\rho(\bar{\Psi}_{\Omega\otimes\Omega})$ where $\rho(\cdot)$ is the spectral radius and the argument matrices are defined using $F(s_t)$, $\Omega_(s_t)$ and transition probabilities. It follows that $\mu_2(F) = [\rho(\Psi_{F\otimes F})]^{1/2}$ and $\mu_2(\Omega') = [\rho(\bar{\Psi}_{\Omega\otimes\Omega})]^{1/2}$ from Theorem 2.5 of Ogura and Martin (2013).

[10] Henceforth, it is a theoretical possibility that $\tilde{\Omega}_1(s_t) \neq \Omega_1(s_t)$ only under regime-switching. And if such a model does exist, it cannot be determinate in both stability concepts.

We illustrate this point using two economic examples in Section 4. Another important finding is that mean-square stable solutions are always at least as many as bounded solutions. If there is no bounded solution, the model has no or a unique MSS solution. If there exists a unique bounded solution, the model has a unique or multiple MSS solutions. If there are multiple bounded solutions, the model has MSS solutions as well.[11]

Finally, we address the computational procedure. As mentioned in footnote 9, it is fast to compute $\mu_2(F_1)$ and $\mu_2(\Omega_1')$ for mean-square stability as it amounts to computing the spectral radii of $\Omega_1(s_t)$ and $F_1(s_t)$ weighted by transition probabilities. By contrast, the measures for boundedness can be computationally demanding. Proposition 3 helps reduce the burden of computing $\mu_1(\tilde{F}_1)$ and $\mu_\infty(\tilde{\Omega}_1')$ in many situations. Appendix G details a practical and efficient implementation procedure of Proposition 3, including the computational time.

4 Applications

In this section, we first present a simple forward-looking model for which our determinacy conditions can be expressed analytically. We then consider a richer bivariate model of fiscal-monetary policy interactions in which all six cases addressed in Table 1 emerge.

4.1 A Fisherian monetary model

We reexamine the Fisherian model analyzed by Davig and Leeper (2007), which consists of a Fisher equation $i_t = r_t + \mathbb{E}_t[\pi_{t+1}]$ and a simple Markov-switching monetary policy rule, $i_t = \alpha(s_t)\pi_t$ where i, r, π are the nominal interest rate, real interest rate and inflation, respectively. Assume that r_t is an exogenous $i.i.d$ process and Assumptions 1 and 2 hold. This model can be cast into a univariate model as follows.

$$\alpha(s_t)\pi_t = \mathbb{E}_t[\pi_{t+1}] + r_t. \tag{8}$$

In this forward-looking model, $A(s_t) = 1/\alpha(s_t)$. The unique fundamental solution $\pi_t = r_t/\alpha(s_t)$ is both bounded and mean-square stable, thus determinacy boils down to judging non-existence of stable sunspot process subject to $w_t = A(s_t)\mathbb{E}_t w_{t+1}$. In this univariate

[11] We also present this classification graphically in Figure 3.

model, $\mu_1(A)$ and $\mu_2(A)$ can be analytically derived as follows (See Appendix H.1).

$$\mu_1(A) = \rho\left(\begin{bmatrix} \frac{p_{11}}{|\alpha(1)|} & \frac{p_{12}}{|\alpha(1)|} \\ \frac{p_{21}}{|\alpha(2)|} & \frac{p_{22}}{|\alpha(2)|} \end{bmatrix}\right) \text{ and } \mu_2(A) = \rho\left(\begin{bmatrix} \frac{p_{11}}{\alpha(1)^2} & \frac{p_{12}}{\alpha(1)^2} \\ \frac{p_{21}}{\alpha(2)^2} & \frac{p_{22}}{\alpha(2)^2} \end{bmatrix}\right).$$

Figure 1 displays the two determinacy regions $\mu_1(A) \leq 1$ and $\mu_2(A) \leq 1$ with respect to $\alpha(1)$ and $\alpha(2)$ for two alternative set of transition probabilities: when regimes are long-lasting ($p_{11} = p_{22} = 0.9$); when regimes are short-lasting ($p_{11} = p_{22} = 0.5$). The calibration with long-lasting regimes is similar to Davig and Leeper (2007). As a numerical example, we use the example in Farmer et al. (2010a) and consider $\alpha(1) = 2$ and $\alpha(2) = 0.93$, which is denoted by $\times$ in Figure 1. We find that $\mu_1(A) = 0.9779 < 1 < \mu_2(A) = 1.0218$. Thus, according to Proposition 1, there exists a mean-square stable sunspot process, but no bounded sunspot exists. That is, there exists a (continuum of) mean-square stable sunspot process $w_t = \Lambda^*(s_{t-1}, s_t) + \eta_t$ such that $\mu_2(\Lambda^{*\prime}) = \frac{1}{\mu_2(A)} < 1$ (See footnote 7). By contrast, $\mu_\infty(\Lambda') \geq \frac{1}{\mu_1(A)} > 1$ among all sunspot processes satisfying $w_t = A(s_t)\mathbb{E}_t[w_{t+1}] = A(s_t)\mathbb{E}_t[\Lambda(s_t, s_{t+1})]w_t$ from Equations (3, 5). Hence, all sunspot processes are not bounded.[12]

This example confirms that boundedness is less stringent than mean-square stability in characterizing determinacy for purely forward-looking models. More concretely, a central bank that wants to rule out mean-square stable inflation processes needs to react more strongly (higher α_2 for a given α_1) or to remain more often in the active regime (higher p_{22}) compared to the case in which it wants to rule out bounded inflation only. The difference between the two determinacy frontiers is quantitatively negligible when regimes are persistent ($p_{11} = p_{22} = 0.9$), but less so when regimes are less persistent ($p_{11} = p_{22} = 0.5$). When regimes are more short-lived, the degree of aggressiveness required in regime 2 to ensure uniqueness of a bounded solution is significantly lower than for mean-square-stability.

4.2 Fiscal-monetary interaction

Our second illustration relies on the model presented in Cho (2021), a simplified version of Leeper (1991) with a regime-switching in monetary and fiscal policies, with only two different

[12] This can also be confirmed numerically as follows. Among all $\Lambda(\cdot)$, we compute $\Lambda^*(\cdot)$ such that $\Lambda^*(\cdot) = \min(\mu_2(\Lambda'))$ following Cho (2021). Specifically it is given by $\Lambda^*(1,1) = 0.4789$, $\Lambda^*(1,2) = 15.6901$, $\Lambda^*(2,1) = 0.0314$, and $\Lambda^*(2,2) = 1.0298$. Then, from footnote 7, $\mu_2(\Lambda^*) = \frac{1}{\mu_2(A)} < 1$. This sunspot process is unbounded because if the economy stays too long in regime 2, inflation diverges as $\Lambda(2,2) > 1$. But since this event is of sufficiently low probability, inflation is still mean-square stable.

regimes.[13] The model consists of the two following equations:

$$\pi_t = \frac{1}{\alpha(s_t)}\mathbb{E}_t\pi_{t+1} + \frac{1}{\alpha(s_t)}r_t, \tag{9}$$

$$b_t = \theta(s_t)b_{t-1} - c(s_t)\pi_t. \tag{10}$$

where b_t is the government debt-to-GDP ratio. Equation (9) is just a rewriting of Equation (8), and Equation (10) is a standard representation of the fiscal block in the spirit of the fiscal theory of the price level. The linearized government budget constraint is $b_t = (1/\beta)b_{t-1} - \tau_t - \bar{b}(1/\beta - \alpha(s_t))\pi_t$ where β is the time discount factor, $\bar{b}$ is the steady state value of b_t. τ_t represents the regime-switching tax policy such that $\tau_t = \delta(s_t)b_{t-1}$. Plugging the tax policy into the budget constraint yields (10) where $\theta(s_t) = 1/\beta - \delta(s_t)$ and $c(s_t) = \bar{b}(1/\beta - \alpha(s_t))$. We follow the convention that the monetary policy is active (passive) if $\alpha \geq (<)1$ and the fiscal policy is active (passive) if $\theta \geq (<)1$ in the fixed regime context. In what follows, we fix the parameter values as $\beta = 0.99$, $\bar{b} = 1$, $p_{11} = 0.95$, $p_{22} = 0.9$. We consider three different scenarios. This model, its calibration and different scenarios are discussed in Cho (2021) and we refer to this paper for complements. See Appendix I for the computational details and further discussions[14].

Regime-switching in Monetary Policy only Suppose that fiscal policy is passive in both regimes with $\theta = 0.8$. We evaluate model determinacy with respect to $\alpha(s_t)$. The MOD solution evolves independently of the fiscal policy and is mean-square stable and bounded as $\mu_2(\Omega_1) = \mu_\infty(\Omega_1) = \theta < 1$. Possible configurations exactly coincide with the Fisherian model. Panel A of Figure 2 shows determinacy frontiers.

Regime-switching in Fiscal Policy only Suppose now that monetary policy is active in both regimes with $\alpha = 1.5$. Determinacy is then examined in terms of fiscal policy $\theta(s_t)$. In this case, all sunspot processes are unbounded and not mean-square stable: $\mu_2(F_1) = \mu_1(F_1) = 1/\alpha < 1$. Therefore, the model is either determinate in boundedness (mean-square stability) if $\mu_1(\Omega_1) < 1$ ($\mu_2(\Omega_1) < 1$), or has no stable solution otherwise. A unique bounded solution requires the fiscal policy to be both passive as in the fixed regime case, whereas a

[13] An application of this type of model in the mean-square stability sense is analyzed by Cho and Moreno (forthcoming), who focus on the economy switching over the zero lower bound and standard regimes.

[14] This example is a good laboratory to compare the computational cost between boundedness and mean-square stability. The average-time is $10 - 100$ larger for boundedness with respect to mean-square stability. Still, for any set of parameters' values, checking determinacy for boundedness takes at most a few seconds (see Appendix G for the details).

temporarily active fiscal policy is admissible for unique mean-square stable solution as long as the policy is not too passive. See Panel B of Figure 2. In this example, determinacy region in boundedness is tighter than that in the mean-square stability, contrary to the case above.

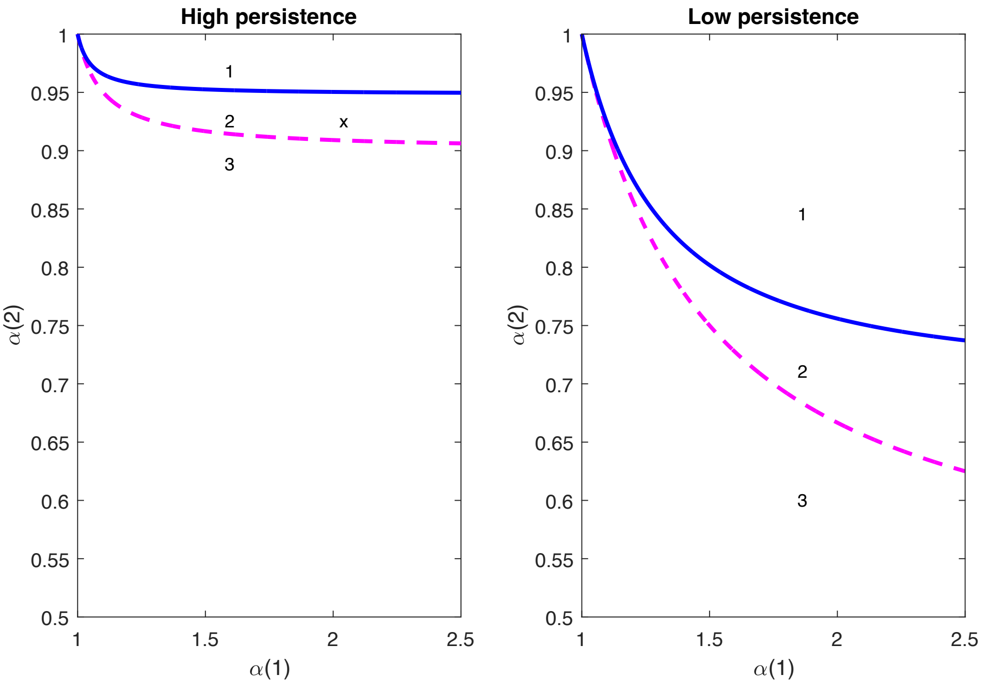
Regime-switching in both policies The last scenario allows both policies to be regime-switching, which is rich enough to exhibit all six partitions of Proposition 3. We fix a policy combination in regime 1 with $\alpha(1) = 1.5$ and $\theta(1) = 0.8$ – an active monetary and passive fiscal policy mix –, and evaluate determinacy in terms of $\alpha(2)$ and $\theta(2)$. Panel C of Figure 2 displays all six partitions. This example suggests that ruling out equilibrium multiplicity requires a more active monetary policy for MSS compared to boundedness, while existence of stable solution requires a more passive fiscal policy for boundedness compared to MSS.

To summarize, our examples clearly demonstrate that the conclusions on model determinacy differ in the two stability concepts when regime switches. While the difference is not quantitatively sizable in our examples, it is an open question whether the difference can be sizeable in more complex/more realistic models.

5 Conclusion

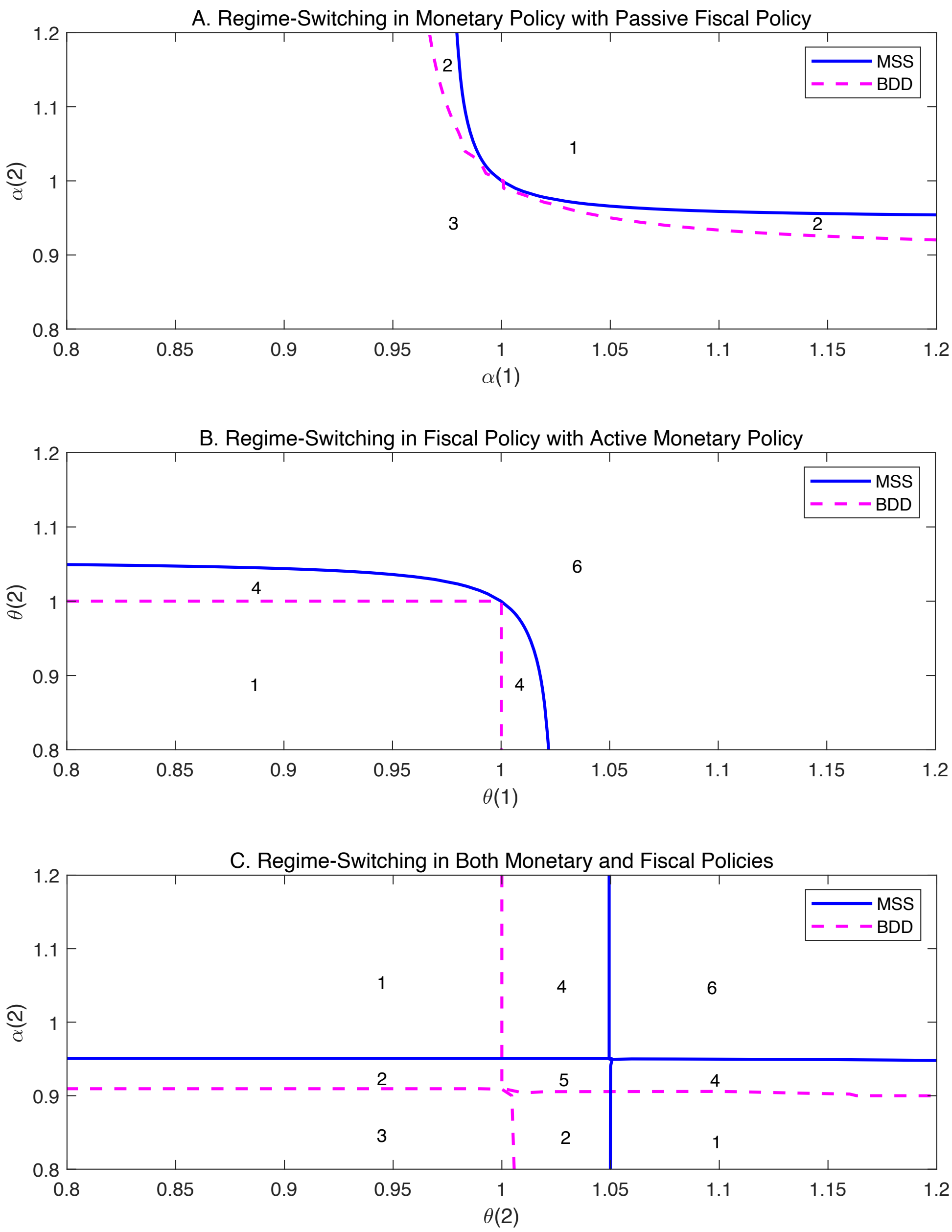
This study proposes a complete methodological foundation for studying the class of macroeconomic models known as Markov-switching rational expectations models. First, we derive necessary and sufficient conditions that ensure a unique, multiple and no stable solution(s) for two standard stability concepts: boundedness and mean-square stability. We characterize these conditions using two simple unified metrics as a function of only one fundamental solution known as MOD solution. Second, we find that only six configurations are possible. The two stability concepts lead to the same conclusion in three of them and they differ in the remaining three. In any case, there are always more mean-square stable solutions than bounded ones in this class of models due to ranking between metrics gauging the stability. We finally believe that our proposed approach is tractable enough and easy to apply for more advanced macroeconomic topics that are hard to analyze in the context of standard linear rational expectations models.

Figure 1: Classification of Regime-switching Fisherian model



This figure depicts the region for determinacy and indeterminacy for the model (8) in boundedness (BDD) and mean-square stability (MSS), for two sets of values of probabilities $p_{11} = p_{22} = 0.9$ (high persistence, left panel), and $p_{11} = p_{22} = 0.5$ (low persistence, right panel). Solid (Dashed) thick line separates the regions of determinacy and indeterminacy in mean-quare stability (boundedness) for $p_{11} = p_{22} = 0.9$. The numbers 1, 2 and 3 represent respectively partition 1: determinacy in both stabilities, partition 2: determinacy in boundedness and indeterminacy in mean-square stability, partition 3: indeterminacy in both stabilities. The point $\times$ corresponds to a set of parameters $\alpha(1) = 2$ and $\alpha(2) = 0.93$ corresponding to a continuum of mean-square stable solutions and a unique bounded solution.

Figure 2: Classification of Regime-switching Model with Monetary and Fiscal Policy Interactions



This figure depicts the region of determinacy, indeterminacy and no stable solution for the model of (9) and (10) in boundedness and mean-square stability under three scenarios. In all panels, solid (Dashed) line represents classifications by mean-quare stability (boundedness). Both stability concepts lead to the same conclusion on the classification for the three regions denoted by 1 (determinacy), 3 (indeterminacy) and 6 (no stable solution). They differ in the remaining three regions denoted by 2, 4 and 5. Refer to the six partitions in Table 1 for more details.

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A More general class of models

The class of MSRE models we considered is general enough to encompass seemingly more general models. First, models with any finite number of leads and lags of the endogenous variables can be written in the form of our model (1). Exogenous variables such as $z_t = R(s_t)z_{t-1} + \epsilon_t$ can also be easily accommodated. Second, models in which A depends on future regime s_{t+1} as well as s_t can be rewritten in the form of (1) by redefining the Markov process. Specifically, suppose that the original model is given by $x_t = \mathbb{E}_t[A(s_t, s_{t+1})x_{t+1}] + B(s_t)x_{t-1} + C(s_t)\epsilon_t$. Let $y_t = A(s_{t-1}, s_t)x_t$, $X_t = [x_t' \ y_t']'$ and $\hat{s}_t = (s_{t-1}, s_t)$, Then it is straightforward to rewrite the model in terms of X_t, $\hat{s}_t$ and ϵ_t in the form of (1). Refer to Cho (2021) for more detail.

B Boundedness, Asymptotic Stability, and Mean-square Stability

As explained after Definition 1, we indeed consider bounded solutions that converge to the steady state in the absence of shocks (in control theory these processes are called asymptotically stable).

Definition 3. *Consider a bounded process*

$$Y_t = H(s_t)Y_{t-1} + G(s_t)\eta_t$$

where s_t is Markovian, with transition matrix P.

We say that Y_t is asymptotically stable if, in absence of shocks, for any $\delta > 0$, for any initial value Y_0 and draws s^t, there exists T such that

$$\forall t \geq T, \qquad \|Y_t\| \leq \delta \|Y_0\|$$

The following result shows that an asymptotically stable process is mean-square stable.

Lemma 2. *Let Y_t be an asymptotically stable process. Then Y_t is mean-square stable.*

Proof. The proof is quite standard in control theory. An asymptotically stable process is exponentially convergent. Precisely, assume that Y_t is asymptotically stable, then there exists K such that

$$\forall Y, \qquad \|H(s_K)H(s_{K-1})\cdots H(s_1)Y\| \leq \frac{1}{2}\|Y\|$$

Iterating this property, we get that for all $q \in \{2, 3, 4, ...\}$,

$$\|H(s_{Kp})H(s_{Kp-1})\cdots H(s_{K(p-1)+1})\cdots H(s_K)H(s_{K-1})\cdots H(s_1)Y\| \leq \left(\frac{1}{2}\right)^p \|Y\|$$

This implies that Y_t is exponentially decreasing and thus mean-square stable. $\square$

Notice that any solution to model (1) can be rewritten as a recursive process of the form of the definition 3 by combining equations (2) and (5), defining $Y_t = [x_t; x_{t-1}]'$ and generalizing the definition of a regime as the pair $[s_t,\ s_{t-1}]$. See also footnote 5. Any solution that is bounded and converges to a certain steady state in the absence of shocks and sunspots hence satisfies Definition 3. The above Lemma then shows that, any bounded solution is also mean-square stable. This finding is a special case of Lemma 1.

C Proof of Lemma 1

Let us first formally introduce the definition of the measure $\mu_n(M)$. For any given set of $n \times n$ matrices $\{M(s_t)\}_{s_t \in \{1,,...,S\}}$ and the transition probability P of which ij-th element is $p_{ij} = \Pr(s_{t+1} = j | s_t = i)$, $\mu_n(M)$ in (6) – $\mu_n(M, P)$, strictly speaking –, is defined as

$$\mu_n(M) = \lim_{k \to +\infty} \left(\sum_{(i_1, \cdots, i_k) \in \{1, \cdots, S\}^k} p_{i_1 i_2} \cdots p_{i_{k-1} k} \|M(i_1)M(i_2)\cdots M(i_k)\|^n \right)^{1/nk}$$

for $n \geq 1$. The claim $\mu_1(M) \leq \mu_2(M) \leq \mu_\infty(M)$ in Lemma 1 is a direct consequence of Lemma 2.9 of Ogura and Martin (2013). *Q.E.D.*

D Proof of Proposition 1

When $B(s_t) = 0_{n\times n}$ in model (1) for all s_t, $x_t = C(s_t)\varepsilon_t$ is the unique fundamental solution and it is bounded and mean-square stable fundamental solution under Assumption 1. Thus we need a necessary and sufficient condition for non-existence of stable sunspot processes for determinacy.

Proof of Assertion 1. Proposition 1 in Barthélemy and Marx (2019) shows that there exists no bounded sunspot processes if and only if $\mu_1(A) \leq 1$ when $A(s_t)$ is non-singular for all $s_t = 1, ..., S$. Their Proposition 1 holds here because our Assumption 2 defines a subset of the class of sunspot shocks in their paper. Thus, all we need to prove here is that their

Proposition 1 also holds when for any s is $r \leq n$, independent of s where r is the rank of $A(s)$. We thus consider the model

$$w_t = A(s_t)\mathbb{E}_t(w_{t+1})$$

For any s, there exist two orthonormal $n \times r$ matrices $V(s)$ and $U(s)$ such that $V'(s)V(s) = \mathbb{1}_r$ and $U'(s)U(s) = \mathbb{1}_r$ and an invertible $r \times r$ matrix $\Phi(s)$ such that

$$A(s) = V(s)\Phi(s)U'(s),$$

from the singular value decomposition theorem. We define $u_t = U'(s_{t-1})w_t$, which satisfies $u_t = U'(s_{t-1})A(s_t)\mathbb{E}_t(w_{t+1}) = U'(s_{t-1})V(s_t)\Phi(s_t)U'(s_t)\mathbb{E}_t(w_{t+1}) = U'(s_{t-1})V(s_t)\Phi(s_t)\mathbb{E}_t(u_{t+1})$. Thus, it follows that $u_t = \hat{A}(s_t, s_{t-1}))\mathbb{E}_t(u_{t+1})$. The results of Barthélemy and Marx (2019) apply to $\hat{A}(s_t, s_{t-1})$ which is an invertible $r \times r$ matrix. Thus, determinacy depends on $\mu_1(\hat{A})$. We notice that

$$\prod_{k=1}^{p} \hat{A}(s_k, s_{k-1}) = V'(s_0)\prod_{k=0}^{p-1} A(s_k)U(s_p)$$

Since $V(s)$ and $U(s)$ are orthonormal,

$$\|\prod_{k=1}^{p} \hat{A}(s_k, s_{k-1})\| = \|\prod_{k=0}^{p-1} A(s_k)\|$$

We finally get that there exists a unique bounded solution if and only if $\mu_1(\hat{A}) = \mu_1(A) < 1$.

Proof of Assertion 2. In his Proposition 1, Cho (2021) proves that there exists no mean-square stable sunspot process if and only if $\rho(\Psi_(F \otimes F)) \leq 1$ for the set of $n \times n$ matrices $\{F(s_t, s_{t+1})\}_{s_t, s_{t+1} \in \{1,,\ldots,S\}}$ when the sunspot shock η_t is mean-square stable. Our $A(s_t)$ is just a special case of $F(s_t, s_{t+1})$. Also recall that $\mu_2(A) = [\rho(\Psi_(A \otimes A))]^{1/2}$ and our Assumption 2 is a subset of the class of mean-square stable sunspot shocks. Therefore, his Proposition 1 holds in our setup as well. *Q.E.D.*

E Proof of Proposition 2

Proof of Assertion 2 Assertion 2 for mean-square stability holds under Assumptions 1 and 2 because it is a special case of Proposition 3 of Cho (2021). *Q.E.D.*

Proof of Assertion 1 requires us to explain several preliminary steps. We first summarize the proof for Assertion 2 in mean-square stability, then derive new equilibrium properties for boundedness analogous to those for mean-square stability. Finally a formal proof for Assertion 1 is presented.

Suppose that there are H fundamental solutions to our model (1), $\Omega_h(s_t)$. Recall that $F_h(s_t)$ and $\Gamma_h(s_t)$ are uniquely associated with $\Omega_h(s_t)$ defined in (4). Without loss of generality, these solutions are indexed in an increasing order of the measure μ_2 such that $\mu_2(\Omega_1') \leq ... \leq \mu_2(\Omega_H')$. $\Omega_1(s_t)$ is referred to as the MOD solution in the MSS sense.

Proposition 2 of Cho (2021) derives the key properties among $\mu_2(\Omega_h')$ and $\mu_2(F_h)$, which are reproduced here using our measure for ease of comparison with the case of boundedness.

P1. $\mu_2(\Omega_h')\mu_2(F_h) \geq \mu_2(\Omega_1')\mu_2(F_h) \geq 1$ and $\mu_2(\Omega_h')\mu_2(F_1) \geq 1$ for all $h \in \{2, \cdots, H\}$.
P2. $\Omega_1(s_t)$ is the unique mean-square stable real-valued MOD solution if $\mu_2(\Omega_1')\mu_2(F_1) < 1$.

The properties P1 and P2 show that the classification only relies on the behavior of the MOD solution. These results are summarized in Proposition 3 of Cho (2021): model (1) is determinate if and only if $\mu_2(\Omega_1') < 1$ and $\mu_2(F_1) \leq 1$, indeterminate if and only if $\mu_2(\Omega_1') < 1$ and $\mu_2(F_1) > 1$, and has no mean-square stable solution if and only if $\mu_2(\Omega_1') \geq 1$.[15] A final remark is that it is easy to see that a model can never be determinate if $\mu_2(\Omega_1')\mu_2(F_1) \geq 1$. To avoid unnecessary complication, we consider the class of MSRE models in which the identification condition for the MOD solution in P2 holds.[16]

Now we turn to the case of boundedness. In their Proposition 3, Barthélemy and Marx (2019) provide sufficient conditions for the three cases: unique, multiple and no bounded solution(s) using the measure μ_∞ defined in (6). Here we provide a much powerful result, extending their proposition: we derive both necessary and sufficient conditions in boundedness for all three cases. Similarly to Cho (2021), we introduce the concept of MOD solution for boundedness. The same set of fundamental solutions are indexed as $\{\tilde{\Omega}_1(s_t)\}_{(s_t)\in\{1,\ldots,S\}}$ in an increasing order of the measure μ_∞ such that $\mu_\infty(\tilde{\Omega}_1') \leq ... \leq \mu_\infty(\tilde{\Omega}_H')$ where $\{\tilde{\Omega}_1(s_t), ..., \tilde{\Omega}_H(s_t)\}$ is the same as, or a permutation of $\{\Omega_1(s_t), ..., \Omega_H(s_t)\}$. $\tilde{\Omega}_1(s_t)$ is now the MOD solution in boundedness sense.

We now report boundedness counterparts of P1 and P2 as follows.

[15] One of the key equilibrium properties for Markov-switching models is that, contrary to fixed regime case and as emphasized in Cho (2021), uniqueness of a stable fundamental solution does not imply determinacy. Therefore, both uniqueness of a stable fundamental solution and non-existence of stable sunspot components must be jointly examined for determinacy as we do in this paper.

[16] Cho (2021) dubs the set of MSRE models with $\mu_2(\Omega_1')\mu_2(F_1) \geq 1$ determinacy-inadmissible, and shows that the *MOD* solution in this case is exotic in the sense that it is not unique (complex-valued or repeatedly real-valued) or such a model consists of completely decoupled systems with a particular structure.

Proposition 4. *Consider the model 1, then the following holds*

1. $\mu_\infty(\tilde{\Omega}'_h)\mu_1(\tilde{F}_h) \geq \mu_\infty(\tilde{\Omega}'_1)\mu_1(\tilde{F}_h) \geq 1$ *and* $\mu_\infty(\tilde{\Omega}'_h)\mu_1(\tilde{F}_1) \geq 1$ *for all* $h \in \{2, \cdots, H\}$.
2. $\tilde{\Omega}_1(s_t)$ *is the unique bounded real-valued MOD solution if* $\mu_\infty(\tilde{\Omega}'_1)\mu_1(\tilde{F}_1) < 1$.

Proof. The proof mimics the strategy of Appendix C in Cho (2021). Using formula (40) in that Appendix, we know that the spectral radius of the matrix

$$\begin{bmatrix} p_{11}\,[\Omega_h(1)]' \otimes F_k(1) & p_{12}\,[\Omega_h(1)]' \otimes F_k(1) \\ p_{21}\,[\Omega_h(2)]' \otimes F_k(2) & p_{22}\,[\Omega_h(2)]' \otimes F_k(2) \end{bmatrix}$$

is larger than 1, when $h \neq k$. This implies that

$$\sum_{(i_1,\cdots,i_p)\in\{1,\cdots,N\}^p} p_{i_1 i_2}\cdots p_{i_{p-1}i_p} \left|\left| \prod_{j=1}^{p} [\Omega_h]'(i_j) \otimes \prod_{j=1}^{p} F_k(i_j) \right|\right| \geq \rho^p$$

where $\rho \geq 1$ and $p = 1, 2, ..., \infty$. According to Lancaster and Farahat (1972), there exist some norms for which

$$\left|\left| \prod_{j=1}^{p} [\Omega_h]'(i_j) \otimes \prod_{j=1}^{p} F_k(i_j) \right|\right| = \left|\left| \prod_{j=1}^{p} [\Omega_h]'(i_j) \right|\right| \left|\left| \prod_{j=1}^{p} F_k(i_j) \right|\right|$$

Thus,

$$1 \leq \left(\max_{(i_1,i_2,\cdots,i_p)\in\{1,\cdots,N\}^p} \left|\left| \prod_{j=1}^{p} [\Omega_h]'(i_j) \right|\right| \right)^{1/p} \left(\sum_{(i_1,\cdots,i_p)\in\{1,\cdots,N\}^p} p_{i_1 i_2}\cdots p_{i_{p-1}i_p} \left|\left| \prod_{j=1}^{p} F_k(i_j) \right|\right| \right)^{1/p}.$$

This implies that $\mu_\infty(\Omega'_h)\mu_1(F_k) \geq 1$. Since $\tilde{\Omega}_h = \Omega_j$ for some $j = 1, ..., H$, it is also true that $\mu_\infty(\tilde{\Omega}'_h)\mu_1(\tilde{F}_k) \geq 1$ for all $h, k = 1, ..., H$ and $h \neq k$. From Lemma 1, $\mu_\infty(\tilde{\Omega}'_1) \geq \mu_2(\tilde{\Omega}'_1)$. This proves Assertion 1: $\mu_\infty(\tilde{\Omega}'_h)\mu_1(\tilde{F}_h) \geq \mu_\infty(\tilde{\Omega}'_1)\mu_1(\tilde{F}_h) \geq 1$ and $\mu_\infty(\tilde{\Omega}'_h)\mu_1(\tilde{F}_1) \geq 1$ for all $h \in \{2, \cdots, H\}$. Hence, there is only two possibility for $\tilde{\Omega}_1$. If $\mu_\infty(\tilde{\Omega}'_1)\mu_1(\tilde{F}_1) < 1$, it must be the MOD solution in boundedness sense, proving Assertion 2. *Q.E.D.* □

Proof of Assertion 1 The classification result for mean-square stability is identical to Proposition 3 of Cho (2021), which holds under our Assumptions 1 and 2. The classification result for boundedness is similar to Proposition 3 of Barthélemy and Marx (2019). But the conditions for unique, multiple and no bounded solutions are both necessary and sufficient, once we show that when $\mu_\infty(\tilde{\Omega}'_1) \geq 1$, there is no bounded solution under Assumptions 1 and 2.

If $\mu_\infty(\tilde{\Omega}_1') \geq 1$, then, for any $\tilde{\Omega}_h$, $\mu_\infty(\tilde{\Omega}_h') \geq 1$. Let us prove by contradiction that there is no bounded solution. Suppose that there exists a bounded solution, that we denote by x_t. Then, there exists an $\tilde{\Omega}_h(s_t)$ for some h such that x_t can be written as Equation (2), and such that for any history, the realization is bounded. For a given set of sunspots $\{\eta_t\}_{t\geq 0}$, and for any initial conditions (x_{-1}, w_{-1}), sequence of shocks $\{\epsilon_t\}_{t\geq 0}$ and regimes $\{s_t\}_{t\geq 0}$, the corresponding process is bounded. For this initial condition x_{-1}, and the sunspots η_t, we construct a process x_t' with an initial condition x_{-1}', then

$$x_t' - x_t = \tilde{\Omega}_h(s_t)(x_{t-1}' - x_{t-1})$$

First, according to Assumption 2, the sunspots are independent of initial condition x_{-1}'. Second, since $\mu_\infty(\tilde{\Omega}_h') \geq 1$, there exist a sequence of regimes s^t and a direction such that $\prod_{k=0}^{T} \tilde{\Omega}_h(s_k)(x_{-1}' - x_{-1})$ is diverging. By construction, the associated realization x_t' is not bounded, which is a contradiction. Therefore, if $\mu_\infty(\tilde{\Omega}_1') \geq 1$, then there is no bounded solution.

F Proof of Proposition 3

First, consider the case $\Omega_1 = \tilde{\Omega}_1$. From Lemma 1, $\mu_1(F_1) \leq \mu_2(F_1)$ and $\mu_2(\Omega_1') \leq \mu_\infty(\Omega_1')$. Depending on the size of $\mu_1(F_1)$, $\mu_2(F_1)$ and 1, there are three possible partitions. Same is true for $\mu_2(\Omega_1')$, $\mu_\infty(\Omega_1')$ and 1. Hence, there are 9 partitions. Classification for the six partitions are shown in Proposition 2. Therefore, we need to show that the remaining partitions cannot arise. The partition with $\mu_1(F_1) \leq 1 < \mu_2(F_1)$ and $1 \leq \mu_2(\Omega_1') \leq \mu_\infty(\Omega_1')$ implies that $\mu_2(F_1)\mu_2(\Omega_1') > 1$, contradicting that Ω_1 is the MOD solution in mean-square stability. Same is true for the partition with $1 < \mu_1(F_1) \leq \mu_2(F_1)$ and $1 \leq \mu_2(\Omega_1') \leq \mu_\infty(\Omega_1')$. The remaining partition with $1 < \mu_1(F_1) \leq \mu_2(F_1)$ and $\mu_2(\Omega_1') < 1 \leq \mu_\infty(\Omega_1')$ implies that $\mu_1(F_1)\mu_\infty(\Omega_1') > 1$, contradicting that Ω_1 is the MOD solution in boundedness.

Second, suppose that $\Omega_1 \neq \tilde{\Omega}_1$. In this case, the ranking of μ_n with associated with the two MOD solutions is given by:

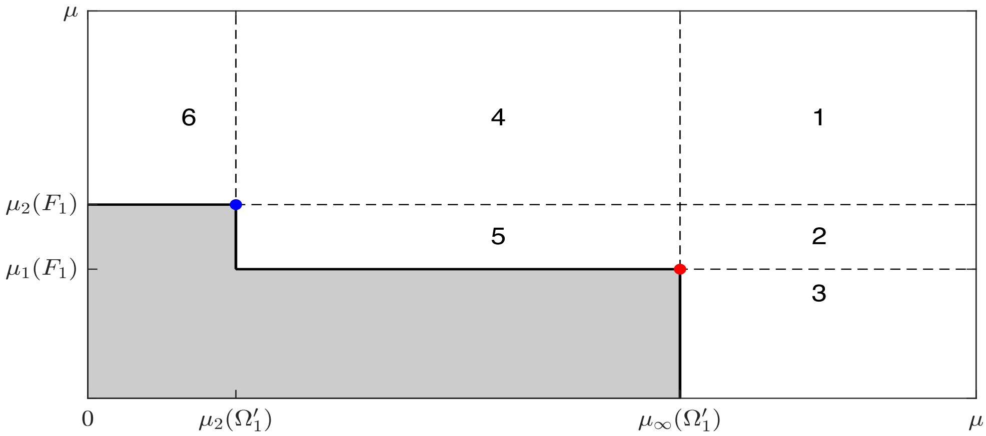
$$\mu_2(\Omega_1') \quad < \quad \mu_2(\tilde{\Omega}_1') \leq \mu_\infty(\tilde{\Omega}_1') < \mu_\infty(\Omega_1') \tag{11}$$

$$\mu_1(\tilde{F}_1) \quad < \quad \mu_1(F_1) \leq \mu_2(F_1) < \mu_2(\tilde{F}_1) \tag{12}$$

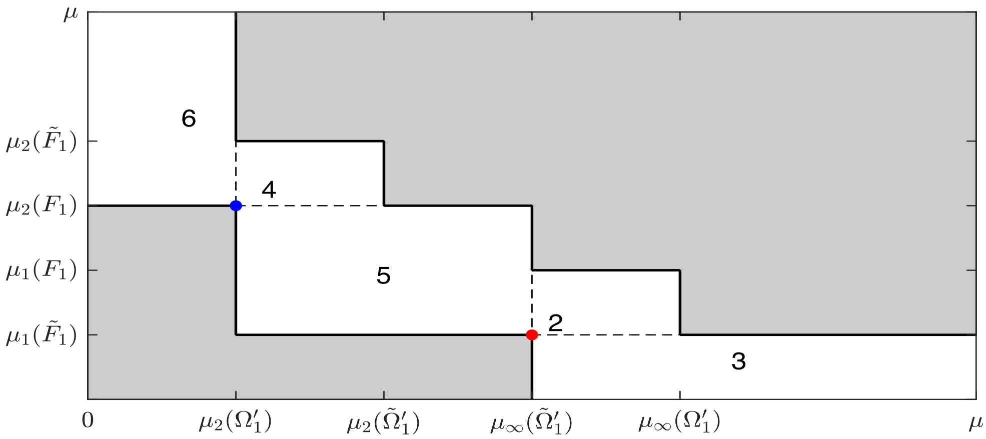
Therefore, $\mu_1(\tilde{F}_1) < \mu_2(F_1)$ and $\mu_2(\Omega_1') < \mu_\infty(\tilde{\Omega}_1')$. Again, there are 9 partitions. Classification for partitions 2 through 6 are shown in Proposition 2, and the remaining three partitions except partition 1 cannot arise by the same logic in the case of $\Omega_1 = \tilde{\Omega}_1$. Now it suffices to show that partition 1, $\mu_1(\tilde{F}_1) < \mu_2(F_1) \leq 1$ and $\mu_2(\Omega_1') < \mu_\infty(\tilde{\Omega}_1') < 1$, cannot arise. In this case, it should be true that $\mu_2(F_1)\mu_2(\tilde{\Omega}_1') < 1$ and $\mu_1(F_1)\mu_\infty(\tilde{\Omega}_1') < 1$ because $\mu_2(\tilde{\Omega}_1') < 1$ and $\mu_1(F_1) < 1$ from (11,12), both of which contradict P1 for mean-square stability and Proposition 4 for boundedness. $Q.E.D.$

Figure 3: Graphical Classification of the MSRE models

Panel A. Classification when $\tilde{\Omega}_1(s_t) = \Omega_1(s_t)$



Panel B. Classification when $\tilde{\Omega}_1(s_t) \neq \Omega_1(s_t)$



This figure displays the classification results summarized in Table 1. Arabic numbers 1 through 6 represent the six partitions in Table 1 in which the coordinate (1,1) can be located. The grey regions surrounded by straight lines are where the point (1,1) cannot be located (bottom left area in Panel A and bottom left and top right areas in Panel B). Panel A displays the classification when $\tilde{\Omega}_1(s_t) = \Omega_1(s_t)$. The Red and blue dot represent the pair $(\mu_2(\Omega_1'), \mu_2(F_1))$ for mean-square stability and $(\mu_\infty(\Omega_1'), \mu_2(F_1))$ for boundedness, respectively: partitions in top right, bottom right and top left regions from the red (blue) point represents determinacy, indeterminacy and no stable solution in mean-square stability (boundedness). Panel B displays the classification when $\tilde{\Omega}_1(s_t) \neq \Omega_1(s_t)$, which is analogous to Panel A except that partition 1 is empty and the blue dot denotes the point $(\mu_\infty(\tilde{\Omega}_1'), \mu_1(\tilde{F}_1))$

G Practical Implications

While Proposition 3 is symmetric for both stability concepts, the numerical assessment of $\mu_\infty(\Omega_1')$ and $\mu_1(F_1)$ for boundedness is computationally more demanding than $\mu_2(\Omega_1')$ and $\mu_2(F_1)$ for mean-square stability, because the latter two are just the function values – spectral radii – of the MOD solution. From a practical point of view, it is thus more efficient to begin with computing the latter two measures for mean-square stability and exploit the ranking information proved in Proposition 3 for boundedness. For instance, suppose that the model is determinate in the mean-square stability (MSS): $\mu_2(\Omega_1') < 1$ and $\mu_2(F_1) \leq 1$. Proposition 3 then implies that $\mu_1(F_1) < 1$ and $\mu_2(\Omega_h') \leq \mu_\infty(\Omega_h')$ for all $h = 2, ..., H$. Henceforth, one has only to check $\mu_\infty(\Omega_h')$ to see that the model has a unique bounded solution if $\mu_\infty(\Omega_h') < 1$ and no bounded solution otherwise. To this end, we propose the following procedure to identify which one of the six partitions a given MSRE models belongs to.

[1] Classification in the mean-square stability: Compute the MOD solution in MSS following Cho (2021). Compute $\mu_2(\Omega_1')$ and $\mu_2(F_1)$. Follow Proposition 3.

[2] Classification in the boundedness: Compute the MOD solution in MSS, and compute $\mu_2(\Omega_1')$ and $\mu_2(F_1)$.

1. If $\mu_2(\Omega_1') \geq 1$, then there is no stable solution in both stability concepts, regardless of $\tilde{\Omega}_1 = \Omega_1$, i.e, the MOD solution in boundedness equals the MOD solution in MSS. (Partition 6)

2. If $\mu_2(\Omega_1') < 1$ and $\mu_2(F_1) \leq 1$ [determinacy in MSS], compute $\mu_\infty(\Omega_1')$.
   (a) If $\mu_\infty(\Omega_1') < 1$, then $\tilde{\Omega}_1 = \Omega_1$ and the model is determinate in boundedness. (Partition 1)

   (b) If $\mu_\infty(\Omega_1') \geq 1$, then $\mu_\infty(\tilde{\Omega}_1') \geq 1$ regardless of $\tilde{\Omega}_1 = \Omega_1$, thus the model has no bounded solution. (Partition 4)

3. If $\mu_2(\Omega_1') < 1$ and $\mu_2(F_1) > 1$ [indeterminacy in MSS], compute $\mu_\infty(\Omega_1')$ and $\mu_1(F_1)$.

   (a) If $\mu_\infty(\Omega_1')\mu_1(F_1) < 1$, then $\tilde{\Omega}_1 = \Omega_1$.

      i. If $\mu_\infty(\Omega_1') \geq 1$, then the model has no bounded solution. (Partition 5)

      ii. If $\mu_\infty(\Omega_1') < 1$ and $\mu_1(F_1) \leq 1$, the model is determinate in boundedness. (Partition 2)

      iii. If $\mu_\infty(\Omega_1') < 1$ and $\mu_1(F_1) > 1$, the model is indeterminate in boundedness. (Partition 3)

(b) If $\mu_\infty(\Omega_1')\mu_1(F_1) \geq 1$, the model cannot be determinate in MSS. Thus, it cannot be determinate in both stabilities. One needs to compute the MOD solution, $\tilde{\Omega}_1$ in boundedness.

   i. If $\mu_\infty(\tilde{\Omega}_1') \geq 1$, then the model has no bounded solution. (Partition 5)

   ii. If $\mu_\infty(\tilde{\Omega}_1') < 1$ and $\mu_1(\tilde{F}_1) \leq 1$, the model is determinate in boundedness. (Partition 2)

   iii. If $\mu_\infty(\tilde{\Omega}_1') < 1$ and $\mu_1(\tilde{F}_1) > 1$, the model is indeterminate in boundedness. (Partition 3)

Two remarks are in order. First, when the model belongs to cases 1 and 2 in this flow chart, one does not need to check whether $\Omega_1(s_t)$ is the MOD solution in boundedness as well and it suffices to compute $\mu_\infty(\Omega_1')$ to identify which partition the model belongs to. For instance, in case 1, $\mu_2(\Omega_1') \geq 1$ implies that $\mu_\infty(\Omega_1') > 1$. Moreover, $\mu_2(F_1) < 1$ because $\mu_2(\Omega_1')\mu_2(F_1) < 1$. Therefore, that $\mu_\infty(\Omega_h') > 1$ for all $h = 2, ..., H$ from P1. Henceforth, all fundamental solutions are unbounded. The same is true for case 2.(a) because $\mu_\infty(\Omega_1') > 1$. Moreover, $\mu_2(F_1) \leq 1$. Second, one needs to identify the MOD solution in boundedness only when the model is indeterminate in mean-square stability. So far, we have never encountered $\tilde{\Omega}_1 \neq \Omega_1$ and hence case 3.(b).

We apply our procedure to the model of monetary and fiscal interaction, using a standard desktop computer, to illustrate the computational time for the parameter values leading to each of the 6 partitions. For mean-square stability, it takes less than 0.001 second to compute the MOD solution and evaluate $\mu_2(\Omega_1')$ and $\mu_2(F)$ in all cases. For boundedness, it is very demanding to compute the exact values of μ_∞ and μ_1. But the main purpose is not to compute those, but to judge whether each of these measures is less than unity or not. In the case of determinacy and no stable solution, it takes much less than a second to draw such a judgment as long as these values are not too close to 1. But it takes several second in the case of indeterminacy. We also tested our procedure for a four-dimensional model of Cho and Moreno (forthcoming) and find that a similar result is obtained although the computational time for implementing our procedure for boundedness increases faster than the increase in the dimension.

H The Fisherian monetary model

H.1 Formula for $\mu_1(A)$ in Section 4.1

Here we show that $\mu_1(A)$ can be easily computed as a spectral radius for the Fisherian model in Section 4.1. This condition is then

$$\lim_{k\to+\infty}\left(\sum_{(i_1,\cdots,i_k)\in\{1,\cdots,2\}_k} p_{i_1i_2}\cdots p_{i_{k-1}i_k}\frac{1}{|\alpha(i_1)||\alpha(i_2)|\cdots|\alpha(i_k)|}\right)^{1/k}$$

$$= \lim_{k\to+\infty}\left(\|\text{vec}(\begin{bmatrix}\frac{p_{11}}{|\alpha(1)|} & \frac{p_{12}}{|\alpha(1)|}\\ \frac{p_{21}}{|\alpha(2)|} & \frac{p_{22}}{|\alpha(2)|}\end{bmatrix}^k)\|_1\right)^{1/k} = \rho\left(\begin{bmatrix}\frac{p_{11}}{|\alpha(1)|} & \frac{p_{12}}{|\alpha(1)|}\\ \frac{p_{21}}{|\alpha(2)|} & \frac{p_{22}}{|\alpha(2)|}\end{bmatrix}.\right)$$

Q.E.D.

H.2 Exploring the existence of bounded sunspots

First,we notice that for any bounded solution,

$$w_t = A(s_t)\mathbb{E}_t w_{t+1}$$

there exists a constant C such that

$$\|w_t\|_\infty < C\lim_{k\to+\infty}\mu_1(A)^k$$

and thus w is zero.

When $\mu_1(A) > 1$, there exist $z(1)$ and $z(2)$ such that

$$\begin{bmatrix} p_{11}A(1) & p_{12}A(1)\\ p_{21}A(2) & p_{22}A(2)\end{bmatrix}\begin{bmatrix} z(1)\\ z(2)\end{bmatrix} = \mu_1(A)\begin{bmatrix} z(1)\\ z(2)\end{bmatrix}$$

The process w_t such that

$$w_t = \mu_1(A)w_{t-1} + z(s_t)\eta_t$$

with $w_0 = z(s_1)$, and η a univariate bounded zero-mean random process, is a solution of the model and is unbounded.

I Model with monetary and fiscal interactions

In this appendix, we provide additional analysis and explain the model in depth. The first step is to compute the MOD solution. For the model consisting of (9) and (10), the state variables are given by b_{t-1} and r_t. By letting $b_t = \phi(s_t)b_{t-1}+q(s_t)r_t$ is a fundamental solution with $\phi(s_t)$ and $q(s_t)$ being unknown coefficients, one can solve for $\Omega(s_t)$ and $F(s_t)$ as follows.

$$\Omega(s_t) = \begin{bmatrix} 0 & \frac{\theta(s_t)-\phi(s_t)}{c(s_t)} \\ 0 & \phi(s_t) \end{bmatrix}, \qquad F(s_t) = \begin{bmatrix} \frac{\phi(s_t)}{\theta(s_t)\alpha(s_t)} & 0 \\ \frac{-c(s_t)\phi(s_t)}{\theta(s_t)\alpha(s_t)} & 0 \end{bmatrix}. \tag{13}$$

In what follows, we will consider three different scenarios. For each scenario, $\phi(s_t)$ and $q(s_t)$ will be derived analytically. Note that, for all scenarios, stability of $\Omega(s_t)$ is governed by the scalar $\phi(s_t)$. We find that the MOD solution in mean-square stability coincides with that in boundedness in all three scenarios. The restriction for the sunspot component $w_t = F(s_t)E_t w_{t+1}$ can also be reduced to a univariate relation such that:

$$w_t^\pi = \frac{\phi(s_t)}{\alpha(s_t)\theta(s_t)}\mathbb{E}_t[w_{t+1}^\pi],$$

where $w_t = [w_t^\pi \ w_t^b]'$. Specifically, the following holds:

$$\prod_{i=1\cdots K} F(s_i) = \begin{bmatrix} \prod_{i=1\cdots K} \frac{\phi(s_i)}{\alpha(s_i)\theta(s_i)} & 0 \\ * & 0 \end{bmatrix}, \qquad \prod_{i=1\cdots K} \Omega(s_i) = \begin{bmatrix} 0 & * \\ 0 & \prod_{i=1\cdots K} \phi(s_i) \end{bmatrix} \tag{14}$$

where $*$ denotes the unimportant off-diagonal terms. Then $\mu_\infty(\Omega')$ is reduced as simple conditions on $\phi(1)$ and $\phi(2)$, while $\mu_1(F)$ depends on $\frac{\alpha(i)\theta(i)}{\phi(i)}$ for $i = 1, 2$.

I.1 Regime-switching in Monetary Policy only

Suppose that fiscal policy is passive in both regimes with $\theta = 0.8$. We find that for this model to be determinate, $\alpha(s_t) > \theta$. Then $\phi(s_t)$ is simply θ, and the MOD solution and the associated $F_1(s_t)$ in Equation (13) are given by:

$$\Omega_1(s_t) = \begin{bmatrix} 0 & 0 \\ 0 & \theta \end{bmatrix}, \ F_1(s_t) = \begin{bmatrix} \frac{1}{\alpha(s_t)} & 0 \\ -\frac{c(s_t)}{\alpha(s_t)} & 0 \end{bmatrix}. \tag{15}$$

This solution is monetary in the sense that inflation evolves independently of the fiscal policy. The model determinacy with $\alpha(s_t)$ can be evaluated following the procedure described in

G. The MOD solution is mean-square stable as $\mu_2(\Omega_1') = \theta < 1$. Since this is regime-independent, $\mu_\infty(\Omega_1') = \theta < 1$ as well, implying that it is also bounded. Therefore, the model is determinate in mean-square stability if $\mu_2(F_1) \leq 1$, and indeterminate otherwise. When $\mu_2(F_1) \leq 1$, the model is also determinate in boundedness as well because $\mu_1(F_1) \leq 1$ from Lemma 1. Hence, we need to compute $\mu_1(F_1)$ only when $\mu_2(F_1) > 1$. Panel A of Figure 2 displays the partition of the parameter space by the two stability concepts. Just like Figure 1, we have partitions 1 (both determinate), 2 (determinate in boundedness and indeterminate in mean-square stability) and 3 (both indeterminate).These results are not surprising and the possible configurations coincide with the Fisherian model. Even if the general model is more general, the fiscal block is always stationary, hence determinacy solely comes from the purely forward-looking monetary block.

I.2 Regime-switching in Fiscal Policy only

The second scenario is one in which monetary policy is active in both regimes with $\alpha = 1.5$. Determinacy is then examined in terms of fiscal policy stance $\theta(1)$ and $\theta(2)$. For this model to contain determinacy region, we find that $\phi(s_t) = \theta(s_t)$ and the MOD solution is given by:

$$\Omega_1(s_t) = \begin{bmatrix} 0 & 0 \\ 0 & \theta(s_t) \end{bmatrix}, \; F_1(s_t) = \begin{bmatrix} \frac{1}{\alpha} & 0 \\ -\frac{c}{\alpha} & 0 \end{bmatrix}. \tag{16}$$

Once again the MOD solution is monetary and inflation does not depend on fiscal policy in the MOD solution, although the solution now depends on the regime-switching fiscal policy. Notice first that $\mu_2(F_1) = 1/\alpha < 1$ because F_1 matrix is now regime-independent. Therefore, the model is determinate in mean-square stability if and only if $\mu_2(\Omega_1') < 1$, and has no stable solution otherwise. To evaluate determinacy in boundedness, it is obvious that $\mu_1(F_1) = 1/\alpha < 1$ as well. Then the model has a unique bounded solution if and only if $\mu_\infty(\Omega_1') < 1$ and has no bounded solution otherwise. In this scenario, we also have three partitions but following the second row in Table 1: 1 (determinacy in both stabilities), 4 (no bounded solution but determinacy in mean-square stability), and 6 (no stable solution in both stabilities). These partitions are displayed in Panel B of Figure 2.

Therefore, in this specification, we have exactly the opposite relation between determinacy in both stabilities, unique boundedness implies unique mean-square stable solution, but the converse is not true. In fact, unique bounded solution requires the fiscal policy to be both passive as in the fixed regime case, whereas a temporarily active fiscal policy is admissible for unique mean-square stable solution as long as the stance is not too strong. Recall that in the first scenario and purely forward-looking models, determinacy in mean-square stability implies determinacy in boundedness. To conclude, determinacy in boundedness is neither necessary nor sufficient for determinacy in mean-square stability in general MSRE models.

I.3 Regime-switching in both policies

The last scenario allows both policies to be regime-switching, which is rich enough to exhibit all six partitions of the classification of the model. In this case, we can evaluate the model partition in terms of four parameters $\alpha(1)$, $\alpha(2)$, $\theta(1)$ and $\theta(2)$, As an illustrative purpose, we fix a policy combination in regime 1 with $\alpha(1) = 1.5$ and $\theta(1) = 0.8$ – an active monetary and passive fiscal policy mix –, and evaluate determinacy in terms of $\alpha(2)$ and $\theta(2)$.

While it is easy to compute the MOD solution numerically, it is instructive to understand its equilibrium properties as follows. Depending on the policy stances in this scenario, the MOD solution can be of the monetary equilibrium similar to (15) and (16), but it can also be fiscal in the sense inflation can be affected by the fiscal block, which would prevail if the fiscal policy is relatively more aggressive than the monetary policy. To be more specific, the solution restriction (4) can be represented as $\phi(j)$, $j = 1, 2$ for the following quadratic polynomial equations:

$$\left[\left(p_{j1}\frac{\phi(1)}{c(1)} + p_{j2}\frac{\phi(2)}{c(2)}\right) - \left(p_{j1}\frac{\theta(1)}{c(1)} + p_{j2}\frac{\theta(2)}{c(2)} + \frac{\alpha(j)}{c(j)}\right)\right]\phi(j) + \frac{\alpha(j)\theta(j)}{c(j)} = 0. \tag{17}$$

Then it is straightforward to show that $\Omega(s_t)$ and the associated $F(s_t)$ satisfy

$$\Omega(s_t) = \begin{bmatrix} 0 & \frac{\theta(s_t)-\phi(s_t)}{c(s_t)} \\ 0 & \phi(s_t) \end{bmatrix}, \qquad F(s_t) = \begin{bmatrix} \frac{\phi(s_t)}{\theta(s_t)\alpha(s_t)} & 0 \\ \frac{-c(s_t)\phi(s_t)}{\theta(s_t)\alpha(s_t)} & 0 \end{bmatrix}. \tag{18}$$

Now we follow the procedure of identifying the six partitions. We first identify the MOD solution, and compute $\mu_2(\Omega_1')$ and $\mu_2(F_1)$. From Equation (17), it is easy to see that there are four solutions for $\phi(s_t)$, thus for $\Omega(s_t)$ as well. In principle, one can identify the MOD solution by computing and finding the smallest $\mu_2(\Omega_h'$, for $h = 1, 2, 3, 4$. But it is difficult to solve for all of the fundamental solutions. Thus one can apply the solution method proposed by Cho (2021) and use the identification condition: If $\mu_2(\Omega)\mu_2(F) < 1$, it is the MOD solution in the mean-square stability sense. The same is true for boundedness: Ω is MOD solution in boundedness if $\mu_\infty(\Omega)\mu_1(F) < 1$. In any case, we find that the MOD solution in boundedness is the same as that in mean-square stability.

Notice that the stability of $\Omega(s_t)$ is governed by the regime-switching scalar $\phi(s_t)$ and the non-existence of stable sunspots and other fundamental solutions can be fully identified by $\frac{\phi(s_t)}{\theta(s_t)\alpha(s_t)}$ in both stability concepts. This simple structure allows us to compute the $\mu_\infty(\Omega_1')$ and $\mu_1(F_1)$ easily. Panel C of Figure 2 displays all six partitions.

Both the top left and bottom right regions belong to partition 1: determinacy in both stability concepts. However, the equilibrium property differs across the two regions. The former is associated with a combination between active monetary and passive fiscal policy in the regime-switching context, whereas the latter is associated with a passive monetary and active fiscal policy combination. Therefore, the equilibrium is monetary in the former and fiscal in the latter. This is qualitatively similar to the fixed regime case, but our contribution is to identify these regions exactly under the both stability concepts.

Region 3 is also similar to the case of passive monetary and passive fiscal policy mix, thus indeterminacy prevails in both stability concepts. In contrast, there is no stable solution in both stability in the region 6, which corresponds to overall active monetary and active fiscal policy mix. Two regions identified by partition 2 make difference between the two stability concepts. In these regions, the model is viewed as determinate in boundedness but indeterminate in mean-square stability sense. Therefore, while the model has a unique bounded solution, there exists a continuum of mean-square stable solutions as well in these regions. Similarly, the two regions denoted by partition 4 represent the case in which there exists a unique mean-square stable solution, but it is unbounded (no bounded solution exists). Finally, the partition 5 is the case where there exists a continuum of mean-square stable solutions (indeterminacy) but all of them are unbounded.