

Strategic Trading, Welfare and Prices with Futures Contracts¹

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ABSTRACT

Derivatives contracts are designed to improve risk sharing in financial markets, but among them, forwards, futures and swaps often appear redundant with their underlying assets: buying the asset and storing it is equivalent to buying it later. I show that imperfect competition in a dynamic market creates an incompleteness, opening gains from trading futures; but surprisingly, in equilibrium, agents trading these contracts have lower welfare than without futures. To mitigate their price impact, buyers (sellers) of an asset postpone profitable trades, exposing themselves to upward (downward) future spot price movements: buyers (sellers) would like to buy (sell) futures. However, when futures are introduced, traders also want to influence the spot price at futures maturity to increase futures payoff: this leads buyers (sellers) to sell (buy) futures. Moreover, despite the absence of market segmentation that would preclude arbitrage, the futures price can be above or below the spot price.

Keywords: Futures, Imperfect Competition, Inefficiency, Mispricing.

JEL classification: G10, G13, G15

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NON-TECHNICAL SUMMARY

Forward and futures contracts are pervasive in security, currency and commodity markets. These derivative contracts allow traders to buy or sell an underlying asset at a future date at a pre-agreed price: when the future asset price is not predictable, this hedges traders against adverse price movements for future transactions. Swaps are portfolios of futures of different maturities.

Futures are often used by agents who cannot trade today: for instance, a wheat producer wishes to hedge against uncertainty regarding the price at which it will sell its crop before sowing, an international company wishes to hedge against exchange rate risk regarding future income or expenses, etc. Were it possible, agents could as well buy or sell their asset immediately, and they would be perfectly hedged without resorting to futures.

But there are important cases, notably major sovereign bond markets, where futures are traded while agents plausibly can trade the asset immediately. For instance, in safe government bond markets (US, Germany, France, ...), it is possible to trade with relatively limited constraints: it is possible to borrow funds to purchase a bond by putting it as collateral for the lender; many maturities are available to investors having such a preference without having to wait for issuance; and spot markets can absorb very large quantities. Yet futures are massively traded: in the US, there have always been between 1.5 and 2 trillion dollars of Treasury futures outstanding over the last three years. One partial explanation is that some investors exploit the spread between futures price and spot prices: but this does not explain why futures are traded in the first place.

Therefore, if it is easy to buy or sell a sovereign bond immediately, why should investors use futures contracts? Is their introduction desirable?

This paper provides a theoretical explanation for this, and finds in particular that 1) futures contracts *decreases* traders' overall profitability, and 2) futures and their underlying asset trade at different prices, although no investor is precluded to trade in either market.

A first result is that a market incompleteness, *i.e.* a situation where traders would like to trade derivatives, is created by imperfect competition. When traders are imperfectly competitive, *i.e.* when they care about the impact of their trades on equilibrium prices, they choose to slice the quantities they want to buy or sell into smaller pieces to be executed successively. This behavior is widely observed in practice. By delaying trade in this way, traders also expose themselves to the risk that the price moves when they trade later: buyers would then fear that the price moves up, and sellers that the price moves down. If futures were introduced, buyers of the underlying asset may be willing to buy futures to sellers of the underlying asset: this rationalizes the use of futures in markets like for safe sovereign bonds.

But when futures are introduced, they are not traded in order to share risk. Traders instead trade futures in a direction opposite to their hedging needs: this is because they also want to influence the futures' payoff, which is the difference between the spot price at futures' maturity, and the futures price at which the contract was set. To do so, they modify their trading strategy in the underlying asset in order to influence the spot price at futures' maturity.

Since futures are not used for risk sharing, they may not smooth traders' revenue and therefore adversely impact their risk-adjusted profitability: I show this formally. However,

this does not necessarily call for further regulation by financial market authorities. In fact, traders in this model are assumed to be large, and all are adversely affected, so that they may devise rules by themselves, *e.g.* within the framework of the futures exchange.

Finally, I show that the futures contract can trade below or above the underlying asset, without assuming storage cost or non-zero interest rates. This is surprising, because an investor could then purchase the cheapest asset (underlying or futures contract) and simultaneously sell the most expensive, locking in a risk-free profit while making both prices converge. Such failures of the law of one price are almost always justified by the fact that some traders are constrained not to trade in one market or the other, and limited capacity by those who can trade in both markets to exploit the difference. For instance, in Keynes' theory of normal backwardation, speculators buy futures to commodity producers and cannot trade in the spot market. But some contemporary commodity traders explicitly refer to simultaneous trade in the spot and futures market as a source of profit. While trading constraints may certainly explain part of observed spreads, imperfect competition in markets with large traders may well be another valid explanation, as this model suggests.

Trading stratégique, profitabilité et prix dans les marchés de contrats à terme

RÉSUMÉ

Les produits dérivés sont conçus pour améliorer le partage du risque au sein des marchés financiers. Toutefois, parmi ceux-là, les contrats à terme de type *forward*, *futures* et *swap* apparaissent redondant avec l'actif sous-jacent : acheter l'actif et le stocker est équivalent à l'acheter à terme. Nous montrons que la concurrence imparfaite dans un marché dynamique crée une incomplétude de marché, ouvrant des gains à l'échange de contrats à terme ; mais de façon surprenante, à l'équilibre, les agents qui échangent ces contrats ont une rentabilité moindre par rapport au cas où ces contrats ne sont pas disponibles. Pour atténuer leur impact sur les prix, les acheteurs (vendeurs) d'un actif retardent une partie de leurs transactions, ce qui les expose à une hausse (baisse) du prix futur : les acheteurs (vendeurs) souhaiteraient donc acheter (vendre) des contrats à terme. Mais lorsque ces contrats sont introduits, les agents souhaitent aussi influencer le prix spot au terme du contrat pour en accroître le rendement : ce qui conduit les acheteurs (vendeurs) à *vendre* (*acheter*) le contrat à terme. De plus, en dépit de l'absence de segmentation du marché qui empêcherait l'arbitrage, le prix du contrat à terme peut être au-dessus ou en-dessous du prix spot contemporain.

Mots-clés : Contrats à terme, concurrence imparfaite, inefficacité, aberrations de prix

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1 Introduction

Forward and futures contracts are pervasive in fixed income, stock, commodities and currency markets. These derivative contracts allow traders to buy or sell an underlying asset at a future date at a pre-agreed price:¹ when the asset price in the future is uncertain, they hedge traders against adverse price movements for future transactions. This hedging demand often comes from an inability to trade the asset directly today: the asset may be a commodity that is not yet produced, short-selling the asset or borrowing funds to purchase the asset may be difficult.²

But there are important examples where futures are massively traded even if the asset can be easily traded today. For instance, US Treasury bonds markets are very liquid, and traders easily borrow funds to buy a bond by putting the bond as collateral, so that few of them should be constrained. Yet the outstanding amount (“open interest”) of Treasury futures peaked above \$2,000bn in February 2020 (CFTC), and was about \$1,600bn in early May 2021. These numbers have been related to arbitrage between Treasury bonds and Treasury futures (Barth and Kahn 2021).³ While arbitrage for sure explains part of the high open interests mentioned above, it does not explain why futures are traded in the first place, if trading is relatively unconstrained. In addition, arbitrage opportunities, usually explained with constraints that prevent traders to act in both markets, in this case deserve additional explanation.

Another concern with futures is that some traders want to manipulate, as evidenced by numerous cases throughout history:⁴ as the futures payoff is the difference between the underlying asset price tomorrow and the pre-agreed futures price, a trader holding futures may be tempted to trade the underlying asset to raise the underlying price tomorrow.⁵

Why do traders trade forwards and futures when constraints to trade on the spot

¹Forward contracts are traded over-the-counter, futures are listed. In this paper, I make no difference between the two types of contracts.

²see Stoll (1979), Hirshleifer (1990), Froot et al. (1993), Raab and Schwager (1993), Routledge et al. (2000) among others.

³For instance, if the former are cheaper than the latter, it is profitable to buy the Treasury bond and enter a short position in the associated futures, then hold the positions until futures maturity. As variations in Treasury bonds prices are offset by an opposite futures payoff, this strategy would be riskless without constraints on arbitrageurs.

⁴In the US, the Commodity Exchange Act of 1936 already forbade futures manipulation. Public disclosure of traders’ position concentration has been done since 1927 by the CFTC and its predecessors, which aims at limiting manipulation. More recently, LIBOR manipulation by bank traders benefited banks’ futures positions.

⁵Kumar and Seppi (1992), Pirrong (1993), Jarrow (1994).

market are light? How do arbitrage opportunities emerge without such constraints? What is the effect of futures on traders' welfare?

To answer these questions, in this paper I provide a dynamic trading model where the only friction is imperfect competition. Traders care about the price impact of their trades, implying that without futures, they choose to defer some profitable trades to tomorrow, and thus to be exposed to asset price risk tomorrow: buyers fear price increases, sellers fear price increases, an incompleteness that futures would solve. But when futures are introduced, traders also want to influence futures payoff by trading the underlying asset at maturity to increase their payoff from the futures. In equilibrium, traders choose *negative* hedge ratios: sellers of the underlying asset buy the futures to buyers of the underlying asset. But both buyers and sellers attempt to influence prices in opposite directions, and prices remain unchanged. Futures are *less* traded if the price volatility *increases*. Overall, futures decrease all traders' welfare unless futures payoff influence is impeded. In addition, futures and the underlying asset trade at different prices (whether or not payoff influence is impeded), although markets are not segmented. I also show that sellers of the underlying asset have greater welfare than buyers when the futures price is higher than the spot price, suggesting that traders in this model trade in the opposite direction to arbitrageurs: my model thus gives an account of deviations from the law of one price in the absence of constraints.

I study a model in which risk averse traders can trade a risky asset at two dates 0 and 1. All information is symmetric. Traders differ by their initial inventories of the asset, and have identical preferences. At each trading date, traders meet in a centralized market. At date 1, some exogenous customers post an inelastic quantity that is unknown at date 0 and independent from the asset payoff. Date 1 price is low when customers sell, and vice versa. Imperfect competition means that traders care about the impact of their trades on equilibrium prices.

I first study the equilibrium without futures: this allows to isolate a hedging demand for futures contracts, on top of being a natural benchmark. At date 0, trades affect both date 0 and date-1 prices, and that there is a trade-off between the two. To see it, consider a seller of the asset. Given other traders' strategies, if they sell more of the asset at date 0, the date-0 price decreases more, which is costly to them, but they arrive at date 1 with less inventory, which makes date-1 price decrease less. Balancing the two impacts, sellers optimally chose to defer some trades to date 1. Buyers make the same reasoning to limit *upward* price pressure. In equilibrium neither sellers nor buyers succeed in moving prices. When competition

increases, traders delay less trade as their impacts on prices are diluted.

A second observation is that the liquidity shock moves date-1 price and thus affects the terms of trade between buyers and sellers. Buyers face the risk of a higher date-1 price than expected, and seller face the risk of lower price. The cost associated with this risk may exceed the price impact benefit of delaying trade. Hence with higher uncertainty on the liquidity shock, traders trade more at date 0.

These two observations shed light on the incompleteness that futures contracts solve: by selling a futures contract, a seller of the underlying asset would hedge against downward date-1 price movements and thus is more willing to postpone trade. Symmetrically, buyers of the underlying are willing to buy futures.

Then I introduce futures contracts, maturing at date 1: these contracts pay off the difference between date-1 price and the futures price, the latter being determined in equilibrium at date 0. With futures, the incentive to affect date-1 price completely changes: on top of mitigating the overall price impact of inventory liquidation, buyers of the future want to push the date-1 price up, and sellers of the futures want to pull it down. This would be achieved by trading the asset at date 1 in the appropriate direction. This incentive has long been fought by public authorities.⁶

I first observe that the incentive to influence the date-1 spot price is so strong that when date-1 price volatility is low, traders' welfare function is not concave anymore: traders would like to buy an arbitrarily large amount of the underlying asset, and to sell an even higher amount of futures in order to re-sell the asset at date 1. This would push date-1 price down and increase the payoff of the short futures' position. The opposite is also possible: to sell spot and to buy futures at date 0. To find an equilibrium, I have to assume that price volatility is high enough.⁷

When an equilibrium exists, the incentive to influence the spot price indeed dominates the hedging motive for trading futures, and traders choose negative hedge ratios: sellers of the underlying asset buy futures, instead of selling futures if they traded for hedging, and buyers of the underlying sell futures. But traders fail to influence the price in equilibrium: both buyers and sellers push the price in their preferred directions, and the forces balance so that equilibrium prices are not affected. In addition, buyers and sellers trade more quickly the underlying asset: sellers want to sell less at date 1, buyers to buy less, in order not to push the

⁶The Commodity Exchange Act, passed in the US in 1936, forbids manipulation in futures market. For theory, see [Kumar and Seppi \(1992\)](#), [Pirrong \(1993\)](#) among many others.

⁷In practice, financing constraints would limit the quantities traded in the asset and futures.

price in a direction that would decrease futures' payoff. I also show that imperfect competition is an essential ingredient to having futures in this environment: as the number of traders becomes arbitrarily large, traders delay less trades and the quantity of futures shrinks to zero.

Regarding traders' welfare, there are two opposite forces: the acceleration of trading in the underlying asset brings a welfare gain. But trading futures in a way contrary to a hedging strategy entails more risk and a welfare loss. I show that traders' welfare is lower with futures than without, because of traders' attempts to influence futures' payoff.^{8,9}

I also show that futures trade at a spread ("basis") with the underlying asset in equilibrium, that is proportional to expectation of the date 1 supply shock. This basis can go in either direction, as observed in practice. The basis does not emerge because of trading constraints, as is the case in many other models. It emerges because the futures price and the spot price are sensitive to the expectation of the supply shock through different channels, and with a higher sensitivity for the futures price: the futures price reflects an expectation of the date 1 price, and the spot price reflects traders' sensitivity to the surplus of date 1 transaction.

While the existence of a basis would not be surprising in a setting with investors constrained not to trade in one market or the other, it is more so in this setting with no constraint: it seems that traders may profitably benefit from this spread by buying spot and selling the futures if the spot price is below the futures price and vice-versa. In fact, I show that traders are better off when they trade *against* the basis, selling cheap and buying expensive. Since the futures price decreases more with the expectation of the supply shock than the spot price, a futures price above the spot reflects a high spot price, thus improved terms of trade in the underlying asset for sellers. Therefore, my model provides a new source of arbitrage opportunities, compared to settings where some traders have limited capacity to enter the spot or the futures market.

⁸I also show in the online appendix that with contracts whose payoff cannot be influenced, futures contracts would indeed raise welfare: thus in the setting of this paper, futures decrease welfare because each party seeks to influence futures payoff.

⁹Such a situation is reminiscent of signal-jamming models à la Fudenberg and Tirole (1986) and Stein (1989), but in these models, the mechanism is different: crucial is the inability of a principal to observe the unobservability of a strategic agent's action by a principal. The principal correctly anticipates manipulation, but the agent always manipulates because if no manipulation was expected by the principal, he/she would profitably manipulate. In my model, all information is symmetric. Recently, Yang and Zhu (2021) have applied this idea to large traders trying to manipulate the central bank's beliefs about the economy's fundamental to force intervention.

Literature review. Many papers study how derivatives arise in the presence of various constraints, in contrast to the present paper. A first strand studies hedging demand when traders face financing constraints: [Froot et al. \(1993\)](#) show how derivatives emerge when it is costly for a firm to raise external funds. [Raab and Schwager \(1993\)](#) show that futures can be used to complete the market when there are short-selling restrictions. [Goldstein et al. \(2013\)](#) provide a model in which some traders end up with negative hedge ratios, as in the present paper. The mechanism is very different, as it relies on information acquisition and market segmentation. A substantial literature on futures trading in commodities markets also studies the role of constraints on traders. First, in the theory of storage¹⁰ the market as a whole cannot short the commodity beyond existing stocks. [Routledge et al. \(2000\)](#) study its implications for futures pricing. This constraint is absent from my model. Second, in the theory of normal backwardation ([Keynes 1930](#), [Hicks 1940](#), [Hirshleifer 1990](#)), producers are willing to offload the price risk of their future production to risk averse speculators who cannot trade in the spot market. Thus, in these models, traders cannot trade in the spot market together with futures markets. [Gorton et al. \(2012\)](#) put the theories of storage and normal backwardation into a single model and link them to inventories.

A large literature make derivatives emerge because of different traders' preferences. [Oehmke and Zawadowski \(2015, 2016\)](#) show that when swaps have exogenously low transaction costs, the natural holder of derivatives are those with short-term horizon, while long term investors hold the underlying. In my model transaction costs are endogenously low, because forwards have a shorter maturity than the underlying. In [Biais et al. \(2016\)](#) and [Biais et al. \(2019\)](#), some traders are specialized in managing an asset and seek to offload the risk of this asset through a derivative contract. In [Biais et al. \(2019\)](#) differences in preferences imply that derivatives are needed to implement optimal risk sharing.

The closest paper to mine is [Rostek and Yoon \(2021\)](#), in which the authors show that under imperfect competition and without exogenous market segmentation, non-redundant derivative products endogenously emerge, with an impact welfare that can be positive or negative. The mechanisms are very different however: in

¹⁰[Kaldor \(1939\)](#), [Williams and Wright \(1991\)](#), [Deaton and Laroque \(1992, 1996\)](#). [Williams and Wright \(1991\)](#) notice that futures and spot markets are redundant, which is not the case in my model.

Rostek and Yoon (2021), the crucial ingredient is that traders have limited ability to condition demand in one asset on prices of other assets. Derivative products are built as portfolios of, and have the same maturity as the underlying assets (e.g. like CDS), and can generally increase or decrease welfare. In the present paper, traders can condition their demand schedule in one asset on other asset prices, and non-redundant futures arise because they have shorter maturity than the underlying; the unambiguously negative welfare effect arises because futures' payoff depends on the underlying spot price (like options, but unlike CDSs).

The present paper connects to the literature on dynamic trading with imperfectly competitive double auctions. Vayanos (1999), Du and Zhu (2017) and Rostek and Weretka (2015) study dynamic trading strategies without forward contracts. Duffie and Zhu (2017) and Antill and Duffie (2018) explore the ability of size discovery mechanisms to overcome the inefficiency. This paper is to my knowledge the first to make forward/futures contracts emerge in this context.

Derivatives manipulation when traders try to influence the underlying spot price has also triggered extensive research (Easterbrook 1986, Kumar and Seppi 1992, Pirrong 1993, Jarrow 1994, among others). In these papers, one player has market power with respect to competitive traders. I study a more general equilibrium where all traders seek to influence the spot price, and to counter influence from traders on the opposite side.

The paper is organized as follows. Section 2 presents the setting. Section 3 solves the imperfect competition equilibrium without forward contracts, and derives the result that uncertainty on the supply shock accelerates trading. Section 4 highlights the gains from trading the risk on the supply shock. Section 5 introduces forward contracts and solves for the equilibrium. Section 6 gives trader welfare with and without derivatives, and compares them. Section 7 focuses on the spread between spot and futures prices. Section 8 concludes.

2 Setting

There are three dates $t = 0, 1, 2$. There is one risky asset that pays off at $t = 2$ an *ex ante* unknown amount

$$v = v_0 + \epsilon_1 + \epsilon_2$$

per unit, where ϵ_1 and ϵ_2 are independent and normally distributed with mean 0 and respective variances σ_1^2 and σ_2^2 .^[11] At date 1, ϵ_1 is released before any action takes place. All information about ϵ_1 or ϵ_2 is symmetric. It is also possible to borrow and save cash at the risk-free rate normalized to zero.

There are two types of traders. Buyers all start with initial inventory $I_{b,0}$ of the risky asset at date 0, and sellers all start with inventory $I_{s,0} > I_{b,0}$. There are B buyers, and S sellers, with the requirement that $N = B + S \geq 3$.^[12] I denote $\bar{I}_0 = \frac{S}{N}I_{s,0} + \frac{B}{N}I_{b,0}$ the average trader inventory. Inventories are publicly known before the date 0 market opens.

Traders maximize the expected utility of their terminal wealth. Their utility is negative exponential (CARA), with risk aversion parameter γ for both types. Thus gains from trade arise from inventory differences. Traders are forward-looking and fully rational: in particular, they perfectly anticipate at $t = 0$ the date-1 equilibrium and adjust their actions accordingly.

At date 0, traders meet in a centralized market where they simultaneously trade the risky asset and enter futures contracts. Futures contracts pay off

$$v_f = p_1 - f_0, \tag{2.1}$$

where p_1 is the asset price at date 1 and f_0 is the futures price at date 0, both to be determined in equilibrium.^[13] For simplicity, I ignore margin constraints associated with futures.

Markets operate through uniform-price double auctions, as in Kyle (1989). Traders of type $k = b, s$ simultaneously post demand schedules $q_{k,0}(p_0, f_0)$ for the risky asset and $x_k(p_0, f_0)$ for the futures contract, conditional on date 0 information. All traders of type k purchase the same equilibrium quantity $q_{k,0}$ of the underlying asset and en-

¹¹The normality assumption is for analytical tractability. It implies that the payoff can be negative without lower bound, which is not consistent with real world limited liability: one could use truncated normal distributions instead. However, at least for date 1 trade and for small probabilities of negative v lead to results approximately identical with and without lower truncation of the probability distribution.

¹²The condition $N \geq 3$ ensure existence of equilibria in linear strategies. When there are only two traders, Du and Zhu (2017) show existence of equilibria in *non-linear* strategies.

¹³Defining futures in this way suggests settlement is financial rather than physical. Physical settlement means that a trader with a short position in the futures contract deliver the underlying asset at maturity. Financial settlement means that a trader with a short futures position pays the difference between the underlying asset price at futures maturity and the futures price when the position was initiated. In practice financial settlement is quite common, even with contracts labelled “physically settled” or “deliverable”, like metal futures on the London Metal Exchange or the COMEX are hybrid: exchange rulebooks indicate that settlement occurs either by delivery or financially by offsetting a contract with an opposite contract at maturity.

ter the same position x_k in the futures contract.¹⁴ A walrasian auctioneer computes the equilibrium prices p_0^* and f_0^* that clear the asset and futures markets. Futures are in zero-net supply, and traders do not have pre-existing futures positions. Thus the market clearing conditions at date 0 are

$$Bq_{b,0} + Sq_{s,0} = 0, \quad (2.2)$$

$$Bx_b + Sx_s = 0. \quad (2.3)$$

Traders of type k arrive in the date-1 market with inventory $I_{k,1} = I_{k,0} + q_{k,0}$.

At date 1, the market for the risky asset re-opens and there is a liquidity shock Q . The signing convention is that when $Q > 0$, some unmodelled traders are willing to sell the asset to traders of type b and s . Conditional on date 0 information, Q is normally distributed with mean $\mathbb{E}_0[Q]$ and variance σ_Q^2 . In the model I will use mostly the variance of Q/N , which is $\sigma_q^2 = \sigma_Q^2/N^2$. The constant N^2 is a convenient normalization. I also assume that Q is jointly normally distributed with, and independent from ϵ_1 and ϵ_2 : this means that Q is a pure liquidity shock. The date-1 market again operates through a uniform-price double auction. Traders of type k post the same demand schedule $q_{k,1}(p_1)$. In equilibrium, they purchase a quantity $q_{k,1}$ of the asset that satisfies the market clearing condition:

$$Bq_{b,1} + Sq_{s,1} = Q. \quad (2.4)$$

(2.4) pins down the equilibrium price p_1^* , and thus the futures payoff. The terminal wealth of a trader of type k is thus

$$W_k = I_{k,0}v + q_{k,0}(v - p_0) + q_{k,1}(v - p_1) + x_k(p_1 - f_0). \quad (2.5)$$

I look for subgame-perfect Nash equilibria in demand schedules, with symmetric strategies for all traders of a given type. The conditions for equilibrium are described in definitions 1, 2 and 5. Specifically, I look for equilibria where demand schedules are linear, as in most of the literature in a CARA-normal framework.¹⁵ In these equilibria, demand schedules are not constrained to be linear, but a linear demand schedule by trader k is the best response to linear demand schedules by other traders.¹⁶ As there is neither information asymmetry nor uncertainty on

¹⁴When traders are strategic, such equilibria exist and are a natural focus of analysis. I further specify the equilibrium concept when traders are strategic in Sections 3 and 5.

¹⁵Cf. Kyle (1989), Vayanos (1999), Malamud and Rostek (2017) among many others.

¹⁶To the best of my knowledge, the existence of equilibria in nonlinear demand schedules with

supply shocks when traders post their demand schedules, an equilibrium multiplicity problem arises (Klemperer and Meyer 1989). I use the trembling-hand stability criterion to select a unique equilibrium (see Vayanos 1999).¹⁷

3 Equilibrium without futures

In this section I study the benchmark case where traders are non-competitive and there are no futures, constraining $x_k = 0$.¹⁸ The setting is similar to Vayanos (1999) and Rostek and Weretka (2015). This allows to emphasize the intertemporal trade-off that traders face at date 0, which the presence of futures affects. In equilibrium, as buyers and sellers follow symmetric strategies, equilibrium prices are not affected. Yet quantities traded are reduced relative to the competitive benchmark.

The intertemporal trade-off that traders face at date 0 is based on foresight of the date-1 equilibrium, given traders' inventories after date 0 trade. Thus Section 3.1 studies the date-1 equilibrium, which also allows to show how traders manage the impact of their trades on contemporaneous price. Then I present the intertemporal trade-off in Section 3.2.

3.1 Date 1 equilibrium: static price impact management

At date 1, trader k maximizes the certainty equivalent of (2.5), which is well known to be the mean-variance criterion given the normal distribution of ϵ_2 :

$$\widehat{W}_{k,1} = I_{k,0}v_1 + q_{k,0}(v_1 - p_0) + q_{k,1}(v_1 - p_1) - \frac{\gamma\sigma_2^2}{2}(I_{k,1} + q_{k,1})^2, \quad (3.1)$$

where $v_1 = v_0 + \epsilon_1$ is the expected payoff v conditional on date 1 information.

Traders take the impact of their demand on the equilibrium price into account, taking the residual demand curve, which is the sum of all other traders' demand

three traders or more and normally distributed payoffs has not been proven. Equilibria in non-linear strategies have been studied with only two traders and normally distributed payoffs (Du and Zhu 2017), or with three traders but payoffs not normally distributed (Glebkin et al. 2020).

¹⁷An alternative approach, in the spirit of Klemperer and Meyer (1989), would assume that traders face supply shocks Q_0 at date 0 and $Q_1 \equiv Q$ at date 1 that are revealed after traders have posted their demand schedules. Thus it would require introducing an additional supply shock Q_0 , which would complicate notations without additional insight.

¹⁸In appendix A, I solve the competitive equilibrium where traders do not manage the price impact of their trades. Traders realize all gains from trade at date 0: they arrive at date 1 with equal inventories, which is the Pareto allocation. This is because they do not care about the price impact of their trades.

curves, as given. For a given quantity $q_{k,1}$ demanded by a trader of type k , this residual demand curve implies an equilibrium price p_1 , and a marginal increase in the quantity demanded by this trader of type k implies a marginal price impact $\partial p_1 / \partial q_{k,1}$. Differentiating the certainty equivalent of wealth (3.1), the first order condition for a trader of type k is

$$v_1 - p_1 - q_{k,1} \frac{\partial p_1}{\partial q_{k,1}} = \gamma \sigma_2^2 (I_{k,1} + q_{k,1}).$$

I look for equilibria in linear strategies: a trader of type k expects to face a linear residual demand curve, and I denote its constant slope by $1/\lambda_{k,1}$, so that $\partial p_1 / \partial q_{k,1} = \lambda_{k,1}$. Thus trader k 's optimal demand schedule given this residual demand schedule is

$$q_{k,1}^*(p_1, \lambda_{k,1}) = \frac{v_1 - p_1}{\lambda_{k,1} + \gamma \sigma_2^2} - \frac{\gamma \sigma_2^2}{\lambda_{k,1} + \gamma \sigma_2^2} I_{k,1}. \quad (3.2)$$

As all traders follow linear strategies given a linear residual demand curve, summing optimal demand (3.2) over other traders, the residual demand curve that a trader faces has slope

$$\begin{aligned} B \times (\lambda_{b,1} + \gamma \sigma_2^2)^{-1} + (S - 1) \times (\lambda_{s,1} + \gamma \sigma_2^2)^{-1} & \quad \text{for a seller,} \\ (B - 1) \times (\lambda_{b,1} + \gamma \sigma_2^2)^{-1} + S \times (\lambda_{s,1} + \gamma \sigma_2^2)^{-1} & \quad \text{for a buyer.} \end{aligned}$$

Requiring consistency of slopes of the residual demand curve faced by all traders with actual equilibrium schedules leads to the following system of equations:

$$\begin{cases} \lambda_{s,1} &= (B(\lambda_{b,1} + \gamma \sigma_2^2)^{-1} + (S - 1)(\lambda_{s,1} + \gamma \sigma_2^2)^{-1})^{-1} \\ \lambda_{b,1} &= ((B - 1)(\lambda_{b,1} + \gamma \sigma_2^2)^{-1} + S(\lambda_{s,1} + \gamma \sigma_2^2)^{-1})^{-1} \end{cases}. \quad (3.3)$$

I can now formally define the equilibrium of the date 1 market.

Definition 1. Demand schedules $q_{k,1}^n(p_1)$ for $k = b, s$ and a price p_1^n is a date-1 equilibrium if:

- demand schedules $q_{k,1}^n(p_1)$ maximize (3.1), given price impact $\lambda_{k,1}$, i.e. satisfy (3.2);
- price impacts $\lambda_{k,1}$ satisfy (3.3);
- the market clearing condition (2.4) holds.

The system of equations (3.3) is easily solved, which leads to the following proposition.

Proposition 1 (Vayanos (1999), Malamud and Rostek (2017)). *A date-1 equilibrium in linear strategies with imperfect competition exists and is unique. In this equilibrium,*

$$\lambda_{s,1} = \lambda_{b,1} = \frac{\gamma\sigma_2^2}{N-2},$$

so that equilibrium demand schedules are:

$$q_{k,1}^n(p_1) = \frac{N-2}{N-1} \left[\frac{v_1 - p_1}{\gamma\sigma_2^2} - I_{k,1} \right]. \quad (3.4)$$

The equilibrium quantities traded by buyers and sellers are

$$q_{b,1}^n = \frac{N-2}{N-1} \frac{S}{N} (I_{s,1} - I_{b,1}) + \frac{Q}{N} \quad \text{and} \quad q_{s,1}^n = \frac{N-2}{N-1} \frac{B}{N} (I_{b,1} - I_{s,1}) + \frac{Q}{N}. \quad (3.5)$$

The equilibrium price is

$$p_1^n = v_1 - \gamma\sigma_2^2 \left(\bar{I}_1 + \frac{N-1}{N-2} \frac{Q}{N} \right), \quad (3.6)$$

where $\bar{I}_1 = S/N I_{s,1}^n + B/N I_{b,1}^n$ is the date 1 average inventory across traders before date 1 trade.

The quantity traded by trader of type k is reduced by a factor $(N-2)/(N-1)$ with respect to the competitive equilibrium, as shown in equation (3.5). This results in imperfect risk sharing: traders of type k end up with

$$I_{k,1} + q_{k,1}^n = \frac{N-2}{N-1} \bar{I}_1 + \frac{1}{N-1} I_{k,1} + \frac{Q}{N}, \quad (3.7)$$

and thus retains a fraction $1/(N-1)$ of her initial inventory $I_{k,1}$. In the perfect competition benchmark, all traders would end up with equal inventories.

The quantity traded at date 1 by each class of traders depends on date-0 equilibrium trade, since $I_{k,1} = I_{k,0} + q_{k,0}$. I fully solve the equilibrium date-1 trade in Section 3.2

The equilibrium price equals the underlying asset expected payoff conditional on information available at date 1, minus an inventory risk premium which is the risk aversion γ times asset variance σ_2^2 , times inventories. The factor $(N-1)/(N-2) > 1$ in front of the liquidity shock Q reflects imperfect competition markup charged

by traders to provide liquidity.

3.2 Date 0: dynamic and static price impact management

I now analyze the intertemporal trade-off regarding price impact at date 0. At date 0, traders take into account both the direct impact on the contemporaneous price p_0 , and the impact on future price p_1^* and quantity $q_{k,1}^*$.

I first consider the certainty equivalent of wealth for a trader of class k . The certainty equivalent of wealth can be expressed, following Lemma 4 in the appendix, as

$$\widehat{W}_{k,0}^n = V_k + S_{k,0}(q_{k,0}) + \widehat{S}_{k,1}(q_{k,0}), \quad (3.8)$$

where

$$V_k = I_{k,0}v_0 - \frac{\gamma(\sigma_1^2 + \sigma_2^2)}{2}(I_{k,0})^2$$

is the certainty equivalent of trader's wealth, assuming he/she does not trade and holds position $I_{k,0}$ until maturity. $S_{k,0}$ is the share of date 0 surplus that a trader of type k gets, and $\widehat{S}_{k,1}$ is the certainty equivalent of the share of date 1 surplus. The share of the date 0 surplus can be expressed as the expected profit from date 0 transaction, $q_{k,0}(v_0 - p_0)$, plus the variation in inventory holding costs induced by trading the quantity $q_{k,0}$:

$$S_{k,0}(q_{k,0}) = q_{k,0}(v_0 - p_0) - \frac{\gamma}{2}(\sigma_1^2 + \sigma_2^2) [(I_{k,0} + q_{k,0})^2 - I_{k,0}^2]. \quad (3.9)$$

The certainty equivalent of the share of date 1 surplus accruing to a trader of type k has the same form, with an expectation operator and a discount factor $z = \left(1 + \left(\frac{N-1}{N-2}\right)^2 \gamma^2 \sigma_2^2 \sigma_q^2\right)^{-1}$, which is in the interval $(0, 1]$ and decreases with the variance of the supply shock:

$$\widehat{S}_{k,1}(q_{k,0}) = \frac{Nz}{N-2} \mathbb{E}_0 \left[q_{k,1}^*(v_1 - p_1^*) - \gamma \sigma_2^2 \left((I_{k,1} + q_{k,1}^*)^2 - (I_{k,1})^2 \right) \right] - \frac{\ln z}{2\gamma}. \quad (3.10)$$

The term $-\ln z/2\gamma$ is positive, and does not depend on $q_{k,0}$. $\widehat{S}_{k,1}$ depends on $q_{k,0}$ first through the equilibrium quantity $q_{k,1}^*$ and the variation in holding costs, which involve $I_{k,1} = I_{k,0} + q_{k,0}$. It also depends on p_1^* , which itself, given equation (3.6), depends on the trader's own quantity $q_{k,0}$ and other traders' date-0 quantities. Thus

date-0 trade has an impact on date-1 price.¹⁹

The intuition underlying this intertemporal price impact is as follows. First consider a seller. Conditional on other traders' equilibrium trades, a trader who has sold a unit of the asset at date 0 arrives at date 1 with a lower inventory $I_{k,1}$ than if he or she had kept it. This trader thus has a higher demand for the asset as indicated by (3.4): this tends to raise date-1 price, as shown by market clearing (2.2); in addition, an increase in date-1 price tends to increase the quantity the trader is willing to sell as shown by (3.4). Therefore by selling more units at date 0, and given other traders' equilibrium trades, a seller tends to make it more profitable to sell other units of the asset at date 1. But selling at date 0 also mechanically reduces the gains from trade at date 1: thus there is a trade-off between selling at date 0 and selling at date 1. The reasoning is symmetric for buyers.

Importantly, the date-1 price, and thus the date-1 surplus $\widehat{S}_{k,1}$ for a trader of type k , also depend on other traders' equilibrium quantities traded at date 0: p_1^* depends on $\bar{I}_1 = B(I_{b,0} + q_{b,0}) + S(I_{s,0} + q_{s,0})$. When a trader sets his/her demand schedule, other traders' date-0 equilibrium quantities are not realized, so that he/she takes these quantities as given. I denote these other traders' quantities with a superscript e , requiring that in equilibrium, they coincide with actual equilibrium quantities:

$$q_{k,0}^e = q_{k,0}^n(p_0^n) \quad \text{for } k = b, s. \quad (3.11)$$

Maximization of (3.8) with respect to $q_{k,0}$ involves date 0 price impact $\lambda_{k,0} = \partial p_0 / \partial q_{k,0}$ for a trader of type k . The $\lambda_{k,0}$ are solution to the following system of equations, analogous to (3.3):

$$\begin{cases} \lambda_{s,0} &= (B(\lambda_{b,0} + \gamma(\sigma_1^2 + \delta\sigma_2^2))^{-1} + (S-1)(\lambda_{s,0} + \gamma(\sigma_1^2 + \delta\sigma_2^2))^{-1})^{-1} \\ \lambda_{b,0} &= ((B-1)(\lambda_{b,0} + \gamma(\sigma_1^2 + \delta\sigma_2^2))^{-1} + S(\lambda_{s,0} + \gamma(\sigma_1^2 + \delta\sigma_2^2))^{-1})^{-1} \end{cases} \quad (3.12)$$

where $\delta = 1 - \frac{N-2}{N}z \in [0, 1)$. The factor $\sigma_1^2 + \delta\sigma_2^2$ comes from differentiation of both $S_{k,0}$ and $\widehat{S}_{k,1}$ with respect to $q_{k,0}$, which is computed in the appendix.

Definition 2. Demand schedules $q_{k,0}^n(p_0)$ for $k = b, s$ and price p_1^n are a date-0

¹⁹At this stage one may correctly see that with market clearing, $\bar{I}_1 = B/NI_{b,1} + S/NI_{s,1} = B/NI_{b,0} + S/NI_{s,0}$, so that the date-1 price does not depend on $q_{k,0}$ anymore. This is a knife-edge case however: if buyers and sellers had different risk aversions, the equilibrium price would not depend on the average inventory anymore, as shown by Malamud and Rostek (2017), and market clearing would not simplify its expression. Therefore, not applying market clearing directly and keeping this intertemporal price impact seems more robust.

equilibrium if:

- for each trader of type k , $q_{k,0}^n(p_0)$ maximizes (3.8) given price impact $\lambda_{k,0}$ and other traders' equilibrium quantities $q_{k,0}^e$;
- traders' price impacts solve (3.12);
- quantities $q_{k,1}^e$ satisfy (3.11);
- the market clearing condition (2.2) holds.

In the proof of Proposition 2 below, I show that the optimal demand schedule is solution to the first order condition of the maximization of the certainty equivalent of wealth (3.8), and given the solution to (3.12), the optimal demand schedules for a trader of type k is

$$q_{k,0}^n(p_0) = \frac{N-2}{N-1} \left[\frac{v_0 - p_0}{\gamma(\sigma_1^2 + \delta\sigma_2^2)} - I_{k,0} - \frac{N-2}{N-1} \frac{\sigma_2^2}{\sigma_1^2 + \delta\sigma_2^2} \approx \left(\frac{\bar{\gamma}}{\gamma} \mathbb{E}_0 \left[\frac{Q}{N} \right] + \frac{1}{N} \sum_{l \neq k} I_{l,1}^e \right) \right]. \quad (3.13)$$

The factor $(N-2)/(N-1)$ comes from contemporaneous price impact minimization. The first line is analogous to date 1. The second line corresponds to traders' management of date-1 surplus $\widehat{S}_{k,1}$.

Expression (3.13) contains $q_{k,0}^e$ on the right-hand side, which I will equate to $q_{k,0}^n(p_0^n)$ later on. Similarly to the date-1 equilibrium, all terms in the demand schedule are reduced by a factor by $(N-2)/(N-1) < 1$. Plugging (3.13) into the market clearing condition (2.2), and imposing (3.11), I derive the following proposition, the proof being in the appendix.

Proposition 2. *The equilibrium quantities traded at date 0 and date 1 by traders of type k are*

$$q_{k,0}^n = \frac{1}{1+A} q_{k,0}^c, \quad (3.14)$$

$$q_{k,1}^n = \frac{N-2}{N-1} \times \frac{A}{1+A} \times q_{k,0}^c + \frac{Q}{N} \quad (3.15)$$

where $A > 0$ is given in appendix, and $q_{k,0}^c$ is the quantity that would be traded if traders were competitive. $q_{k,0}^c$ is given by

$$q_{b,0}^c = \frac{S}{N}(I_{s,0} - I_{b,0}) \quad \text{and} \quad q_{s,0}^c = \frac{B}{N}(I_{b,0} - I_{s,0})$$

for buyers and sellers respectively. Moreover, $\frac{1}{1+A} \leq \frac{N-2}{N-1}$, and $\frac{1}{1+A}$ converges to one as the number of traders N becomes infinite. The date-0 equilibrium price is:

$$p_0^n = v_0 - \gamma(\sigma_1^2 + \sigma_2^2)\bar{I}_0 - \gamma\sigma_2^2 z \mathbb{E}_0 \left[\frac{Q}{N} \right]. \quad (3.16)$$

A is a rate of demand reduction for date 0. The equilibrium quantity traded $|q_{k,0}^n|$ is lower than the competitive quantity $|q_{k,0}^c|$ since A is positive. The demand reduction factor $1/(1+A)$ is also lower than $(1+(N-2)/(N-1))$, the factor that would prevail if there were no subsequent trading round, because traders care about the impact of their trades on both date-0 and date-1 prices. This is as in [Rostek and Weretka \(2015\)](#). Regarding date-1 quantities, the first term is the fraction $A/(1+A)$ of the competitive quantity that was not traded at date 0, of which a fraction $(N-2)/(N-1)$ is actually traded because of imperfect competition.

4 A market incompleteness that futures could fill

In this section I show that in the equilibrium without futures, there are gains from trading futures, because of the risks over the supply shock Q (subsection [4.1](#)) and over news on terminal payoff ϵ_1 (subsection [4.2](#)).

Then I deduce that this implies sellers of the underlying asset also selling the futures to buyers of the underlying asset. In online appendix [C](#) I show that a theoretical futures whose payoff cannot be influenced, *i.e.* whose payoff is simply a linear combination of ϵ_1 and Q , involves trading the underlying asset and the “futures” in the same direction.

4.1 Imperfect competition creates gains from trading risk over Q

Here I show that imperfect competition creates gains from trading the risk on Q because buyers and sellers then have opposite exposure to a risk on Q . There are two effects that play in the same direction, which show up in traders’ utility after date-1 trade:

$$\widehat{W}_{k,1}^n = I_{k,0}v_1 + q_{k,0}^n(v_1 - p_0^*) - \frac{\gamma\sigma_2^2}{2} (I_{k,1})^2 + S_{k,1}^n \quad (4.1)$$

$$\text{with } S_{k,1}^n = q_{k,1}^n(v_1 - p_1^*) - \frac{\gamma\sigma_2^2}{2} \left[(I_{k,1}^n + q_{k,1}^n)^2 - (I_{k,1}^n)^2 \right]. \quad (4.2)$$

with date-1 equilibrium values of given by (3.6), (3.5) and (3.7). Traders utility depends on Q only through the net surplus $S_{k,1}^n$ from date 1 transaction. $S_{k,1}^n$ is composed of the expected payoff component $q_{k,1}^n(v_1 - p_1^*)$, and the impact on risk holding cost $\frac{\gamma\sigma_2^2}{2} [(I_{k,1}^n + q_{k,1}^n)^2 - (I_{k,1}^n)^2]$. The two effects relate to each of these components.

First effect: price risk affecting the terms of trade. Under imperfect competition, date 0 sellers are still willing to sell at date 1: thus they dislike when customers sell at date 1 because it decreases the price at which they sell, leaving their valuation of the asset unchanged. Symmetrically buyers dislike when customers buy at date 0. This is not the case under perfect competition, because all intertrader gains from trade are exhausted at date 0 and all traders arrive with symmetric inventories at date 1.

Formally, this effect relates to the expected payoff component of date 1 surplus: using equilibrium price (3.6) and quantity (3.5), one sees that

$$q_{b,1}^n(v_1 - p_1^*) = \underbrace{\left(\frac{N-2}{N-1} \frac{S}{N} (I_{s,1}^n - I_{b,1}^n) + \frac{Q}{N} \right)}_{q_{b,1}^n} \underbrace{\gamma\sigma_2^2 \left(\bar{I}_0 + \frac{N-1}{N-2} \frac{Q}{N} \right)}_{v_1 - p_1^*},$$

and symmetrically for sellers. A given realization of Q has three effects on the expected payoff of date 1 transaction. The first is that the quantity Q impacts the terms of trade between buyers and sellers, which is the term $\gamma\sigma_2^2 Q/N \times \frac{S}{N} (I_{s,1}^n - I_{b,1}^n)$: buyers ($I_{s,1}^n - I_{b,1}^n > 0$) make an unexpected profit when customers are sellers ($Q > 0$), while they make an unexpected loss when customers buy at the same time as them ($Q < 0$). By contrast, sellers make a loss when customers sell at the same time as them, and make an unexpected profit when customers buy. This effect is not present under perfect competition, because gains from trade are exhausted at date 1, so that $I_{b,1}^c = I_{s,1}^c$.

Second, Q affects the quantity traded given the price, to which the term $Q/N \times \gamma\sigma_2^2 \bar{I}_0$ corresponds. Both classes of traders are exposed in the same way to this risk, with the same marginal utility: it seems that there is room for trading this part of the risk.

Third, Q has a second order effect represented by $(Q/N)^2$, which is always positive: unexpected customer sales occur at an unexpectedly low price, which increases the surplus both classes of traders earn from trading with customers. Again all traders are exposed in the same way to this risk, with the same marginal utilities

related to this effect.

Overall I conclude that only the price effect leads marginal utilities between buyers and sellers make marginal expected payoff from date 1 transaction differ for each traders.

Second effect: asymmetric effect of Q on holding costs. The quadratic form of risk holding costs $\frac{\gamma\sigma_2^2}{2}(I_{k,1}^* + q_{k,1}^*)^2$, and the fact that all traders get the same share of customer trades, implies that traders with large date 1 initial inventory $I_{k,1}^*$ incur a larger cost (relief) than buying traders when customers sell (buy) than traders with low date 1 initial inventory. Indeed the marginal holding cost for trader k is

$$\frac{\partial \frac{\gamma\sigma_2^2}{2}(I_{k,1}^* + q_{k,1}^*)^2}{\partial Q} = \frac{\gamma\sigma_2^2}{N} \left(\frac{N-2}{N-1} \frac{S}{N} I_{s,1}^{eq} + \frac{1}{N-1} \frac{S}{N} I_{b,1}^* + \frac{Q}{N} \right).$$

As $\frac{N-2}{N-1} < 1$, sellers face a larger marginal cost of customer trades than buyers (since $I_{s,1}^{eq} > I_{b,1}^{eq}$).

The first and the second effect go in the same direction, which leads to the following proposition.

Proposition 3. *The supply shocks generates a risk over the date 1 terms of trade between buyers and sellers, to which both are exposed in an opposite way. When customers sell ($Q > 0$), traders starting date 1 with a higher inventory are marginally worse off than traders with low inventory. The relation is reversed when customers buy. Thus there are gains from trading this risk.*

Thus under imperfect competition where traders still have unequal inventories after date 0 trade, there are gains from traders trading risk on Q .

Under perfect competition, traders have equal inventories after date 0 trade and there are no gains from trading risk on Q .

4.2 Imperfect competition and gains from trading risk on ϵ_1

The uncertainty over ϵ_1 does not affect the terms of trade between traders, because it is common value: following a shock, all traders adjust their demand schedules by the same amount. But there are still gains from trading the risk, which parallels that of trading the underlying asset: because not all gains from trade are realized after date 0 trade, sellers hold too much risk over ϵ_1 between dates 0 and 1,

and buyers carry too little. Therefore, a contract that shares the risk on ϵ_1 would allow to share corresponding holding costs more efficiently.

4.3 How would traders trade futures for hedging purposes

In Section 4.1 I showed that sellers were worse off when Q was higher than expected, and buyers were worse off when Q was lower than expected. This suggests that sellers would like to buy a contract that pays off Q , and that buyers would be willing to do so.

In Section 4.2, I showed that sellers were willing to sell a contract that pays ϵ_1 to buyers, and buyers would be willing to do so.

From (3.6), a futures contract has a gross payoff

$$p_1^* = v_0 - \gamma\sigma_2^2\bar{I}_1 + \epsilon_1 - \frac{N-1}{N-2}\gamma\sigma_2^2\frac{Q}{N}.$$

Therefore a futures is a portfolio of contracts that pay off ϵ_1 and $-Q$, which sellers would like to sell to buyers, and buyers would accept.²⁰ In appendix I solve for equilibrium with a theoretical contract that pays off $\epsilon_1 - \frac{N-1}{N-2}\gamma\sigma_2^2\frac{Q}{N}$, and show that indeed sellers of the underlying also sell this contract to buyers of the underlying asset. As a result, all traders postpone more of their trade to date 1.

Yet a genuine futures contract paying off p_1^* also involves the term

$$-\gamma\sigma_2^2\bar{I}_1 = -\gamma\sigma_2^2\left(\sum_{l \neq k} \frac{I_{l,1}}{N} + \frac{I_{k,0} + q_{k,0}}{N}\right),$$

which trader k wishes to influence: it turns out to completely reverse the price impact trade-off that traders face at date 0, and thus changes the equilibrium. This is the topic of Section 5.

²⁰Although not necessarily in optimal quantities. I also solved the equilibrium with contracts that pay off ϵ_1 and Q separately: equilibrium quantities of each contract involve different proportions than those implied by a futures contract. This is because each component of the price has a different role for traders: Q affects the terms of date 1 trade, and ϵ_1 relates to the payoff that traders get at maturity.

5 Equilibrium trades with futures contracts

In this section, I study the equilibrium with futures contracts and imperfect competition.²¹

5.1 Futures affect intertemporal price impact

The date-0 certainty equivalent of wealth is (see proof in appendix [B.4.1](#))

$$\begin{aligned} \widehat{W}_{k,0}^f(q_{k,0}, x_k) &= V_k + S_{k,0}(q_{k,0}) + \widehat{S}_{k,1}(q_{k,0}) \\ &\quad + (\widehat{p}_1 - f_0)x_k - \gamma(\sigma_1^2 + (1-z)\sigma_2^2)I_{k,1}x_k - \frac{\gamma}{2}\left(\sigma_1^2 + \frac{1-z}{\alpha}\sigma_2^2\right)x_k^2, \end{aligned} \tag{5.1}$$

where \widehat{p}_1 is a risk-adjusted expectation of date-1 price p_1^* :

$$\widehat{p}_1 = v_0 - \gamma\sigma_2^2z\left(\bar{I}_1^e + \frac{N-1}{N-2}\mathbb{E}_0\left[\frac{Q}{N}\right]\right)$$

As in [\(3.8\)](#), V_k is the mean-variance certainty equivalent of wealth with trader k 's initial inventory position $I_{k,0}$, $\widehat{S}_{k,0}$ and $\widehat{S}_{k,1}$ are trader k 's shares in date 0 and date 1 transaction surpluses in the underlying asset. The second line in [\(5.1\)](#) is the risk-adjusted payoff of futures. It consists first in an expected payoff $x_k(\widehat{p}_1 - f_0)$, which depends on the trader's trade. It also includes a hedging term in $I_{k,1}x_k$ (selling futures, $x_k < 0$, hedges against large inventory risk $I_{k,0} > 0$), and a term in x_k^2 that reflects the riskiness of the futures payoff.

Futures contracts payoff is open to influence, which shows up in the term $x_k(\widehat{p}_1 - f_0)$ in [\(5.1\)](#): a trader buying the futures contract would like a high underlying price at date 1, and conversely. As noted in Section [4.3](#), this goes through the term in \bar{I}_1^e . To influence the price, a trader buying futures would like to sell massively at date 1 to make p_1^* increase. In the limit, such an investor could sell the underlying asset at date 0 only to re-purchase it at date 1, provided that the costs of doing so are offset by a higher payoff on the derivative position. A trader selling the futures would like to do the converse.

²¹In appendix [A](#), I solve the competitive benchmark with futures. At date 0, these contracts are perfect substitutes with the underlying asset. Formally, from the first order conditions I cannot deduce well-defined demand schedules for the futures and the underlying asset. This illustrates the redundancy of futures in a perfectly competitive setting: models based on the Theory of Storage with perfect competition ([Williams and Wright 1991](#), [Routledge et al. 2000](#)) also find that futures are redundant with the underlying commodity.

There are two costs of influencing futures payoff in this way. First, the cost of holding the excess or deficit position in the underlying asset and in the futures contract from date 0 to date 1 (σ_1^2 in the model). Second, the uncertainty on date-1 price entailed by uncertainty on the supply shock Q . When these two costs are low, the profit function [5.1](#) is not concave anymore, which means the following^{[22](#)}

Lemma 1. *The certainty equivalent of wealth [5.1](#) is concave if and only if σ_q^2 is above a threshold $\bar{s}(\sigma_1^2)$. This threshold decreases with σ_1^2 .*

When σ_q^2 is below $\bar{s}(\sigma_1^2)$, the certainty equivalent of wealth is not concave: a trader with certainty equivalent of wealth [5.1](#) would try to achieve arbitrarily large profit by submitting an arbitrarily large demand for the underlying asset, and an opposite and even larger demand for the futures.

Traders' strongest incentive is thus not to allocate trades to minimize date 0 and date 1 buying or selling pressure anymore, but to trade in the spot market in order to raise the futures payoff by affecting date-1 price. In what follows, I assume that $\sigma_q^2 > \bar{s}(\sigma_1^2)$.

5.2 Equilibrium definition

I assume imperfect competition in both the underlying asset and futures markets. Therefore, trader k takes the impact of trade in each market on both prices simultaneously. To solve the equilibrium, I apply methods from [Malamud and Ros-tek \(2017\)](#), which extends the method used without futures to a setting with several assets. A 2×2 matrix of price impacts $\Lambda_{k,0}$ replaces the scalar price impact $\lambda_{k,0}$ of Section [3](#), and the equilibrium condition becomes:

$$\begin{cases} \Lambda_{s,0} &= (B(\Lambda_{b,0} + \gamma\Sigma_f)^{-1} + (S - 1)(\Lambda_{s,0} + \gamma\Sigma_f)^{-1})^{-1} \\ \Lambda_{b,0} &= ((B - 1)(\Lambda_{b,0} + \gamma\Sigma_f)^{-1} + S(\Lambda_{s,0} + \gamma\Sigma_f)^{-1})^{-1} \end{cases} \quad (5.2)$$

where the 2×2 matrix Σ_f is given in the appendix. As in Section [3.2](#), given that $\widehat{S}_{k,1}$ depends on p_1^* which itself depends on other traders' date-0 equilibrium trades in the underlying asset, a trader's demand schedules also depend on these quantities, which have to coincide with actual equilibrium quantities.

²²This does not necessarily imply that an equilibrium does not exist in the case above, even if the profit function is not quasi-concave (the graph of $\widehat{W}_{k,0}^f$ is a hyperbolic paraboloid): posting an arbitrarily large demand in both assets also entails arbitrarily large transaction costs at date 0. I leave the question of equilibrium existence to future research.

Definition 3. Demand schedules $q_{k,0}^*(p_0, f_0)$ in the underlying asset and $x_k^*(p_0, f_0)$ in futures contracts, and spot price p_0^* and futures price f_0^* are an equilibrium if:

- demand schedules $q_{k,0}^*(p_0, f_0)$ and $x_k^*(p_0, f_0)$ maximize (5.1);
- price impacts matrices $\Lambda_{b,0}$ and $\Lambda_{s,0}$ satisfy (5.2);
- quantities $q_{k,0}^e$ satisfy 3.11;
- market clearing conditions (2.2) and (2.3) hold.

5.3 The failure of futures payoff manipulation

At date 1, whatever equilibrium trades, the price (3.6) is unchanged by futures trading: p_1^* depends on traders' average date 1 inventory

$$\begin{aligned}\bar{I}_1 &= \frac{S}{N}(I_{s,0} + q_{s,0}) + \frac{B}{N}(I_{b,0} + q_{b,0}) \\ &= \frac{S}{N}I_{s,0} + \frac{B}{N}I_{b,0} \equiv \bar{I}_0,\end{aligned}$$

by market clearing at date 1 (equation (2.4)). One could have applied market clearing to recognize $\bar{I}_1 = \bar{I}_0$ from the beginning, which would have changed the results. But applying market clearing in this way is peculiar to the situation where buyers and sellers share the same risk aversion parameter: with different risk aversions parameters, the date-1 price would be affected by date-0 trades (Malamud and Rostek 2017). The following proposition shows that the underlying date-0 price is also not affected by futures trading. It also derives the futures price.

Proposition 4. The underlying asset price p_0^* is equal to the price without futures:

$$p_0^* = p_0^n. \quad (5.3)$$

The futures price is

$$f_0^* = v_0 - \gamma(\sigma_1^2 + \sigma_2^2)\bar{I}_0 - \frac{N-1}{N-2}\gamma\sigma_2^2 z \mathbb{E}_0 \left[\frac{Q}{N} \right]. \quad (5.4)$$

The proof is in the appendix. There is no inventory risk premium associated with futures, because the contract is in zero net supply. I also show in an online appendix C that with futures whose payoff cannot be influenced, spot and futures prices are exactly the same.

One could expect that more buyers implies more upward price pressure and vice-versa. This is not the case: the underlying price, the futures price and thus the forward-spot spread do not depend on the relative number of buyers and sellers, in spite of efforts of both sides to impact them. Ultimately, this stems from the fact that risk aversions, *i.e.* elasticities, on both sides are the same. Given the static results by [Malamud and Rostek \(2017\)](#), it is likely that prices would depend on the numbers of buyers and sellers with heterogenous risk aversions.

5.4 Equilibrium trades with futures

While the underlying price does not change, equilibrium trading patterns do change with the introduction of futures. The following proposition establishes first that sellers of the underlying asset buy futures, *i.e.* choose negative hedge ratios.

Proposition 5. *If σ_q^2 is sufficiently large with respect to σ_1^2 , an equilibrium with futures exists and is unique. In the equilibrium with futures, sellers of the underlying asset purchase futures to buyers of the underlying:*

$$q_{k,0}^* = \frac{1}{1 + A_f} q_k^c \quad (5.5)$$

$$q_{k,1}^* = \frac{A_f}{1 + A_f} \times \frac{N - 2}{N - 1} \times q_k^c + \frac{Q}{N} \quad (5.6)$$

$$x_k^* = h_f \left(q_{k,1}^* - \frac{Q}{N} \right) \quad (5.7)$$

where h_f is negative. q_k^c is the competitive quantity traded at date 0 ([A.12](#)), and $A_f > 0$ is the date 0 rate of trade delay with futures contract and depends on N , on z and on the ratio σ_1^2/σ_2^2 .

The quantity of futures traded x_k^* is equal to the quantity deferred to date 1, $q_{k,1}^* - Q/N$, times the hedge ratio h_f .²³ As stated in [Section 4](#), if the hedging motive dominated, traders would choose positive hedge ratios, *i.e.* trade futures and the underlying asset in the same direction. But in equilibrium, hedge ratios are negative.

For futures buyers, influencing futures payoff involves raising date-1 price, thus purchasing more or selling less at date 1; and likely preparing this at date 0 by selling more, or purchasing less. The opposite holds for futures sellers. Who should

²³The hedge ratio is to be computed with respect to expected date-1 quantity (minus the supply shock), not to date-0 quantity or date 0 inventory. This general form also holds for my theoretical non-manipulable futures in the online appendix [C](#) except that the hedge ratio is positive.

be futures buyers? If these were underlying buyers, they would carry too little risk on ϵ_1 , and sellers too much, from date 0 to date 1. Thus buyers and sellers all prefer that futures buyers are underlying asset sellers, which is opposite to hedging would require. The following proposition, proven in the appendix, confirms this.

Proposition 6. *Futures accelerate trading in the underlying asset:*

$$\begin{aligned} |q_{k,0}^n| &< |q_{k,0}^*| \\ |q_{k,1}^n| &> |q_{k,1}^*| \end{aligned}$$

This is because the rates of trade delay are such that $A > A_f$.

Influence on date-1 price is more likely to fail if this price is more uncertain. Consider uncertainty associated with Q first: if σ_q^2 increases, trying to raise date-1 price (for a futures seller) is more likely to be offset by date 1 customers pushing the price downward. Thus the payoff influence motive for trading decreases as σ_q^2 increases. Moreover, as stated in Section [5.1](#), another cost of trying to influence futures payoff is the risk on ϵ_1 , which entails a more uncertain futures payoff. The following proposition, proven in the appendix, states that this is reflected in quantities.

Proposition 7. *The quantity of futures traded $|x_k^*|$ decreases as σ_q^2 increases, and shrinks to zero as σ_q^2 diverges to infinity.*

The quantity of futures traded also decreases as σ_1^2 increases.

This proposition thus states that traders trade less futures when they have more hedging needs. The intuition is clear, since when they try to influence futures payoff, traders do not hedge with futures, but choose a naked exposure to date-1 price through the futures contract.

5.5 The perfect competition limit: futures are not traded

Here I show that when the number of traders grows to infinity, futures position and open interest shrink to zero, justifying the claim that imperfect competition creates a demand for futures.

To look at the limit when N grows to infinity, I have to care about whether N grows because one adds an overwhelming proportion of buyers or of sellers, or if some balance between buyers and sellers is kept. Formally, the latter means that the ratio B/S remains finite and stays away from zero.

A second issue is that to isolate the pure effect of imperfect competition, I have to care about adding traders while keeping total inventories constant: for instance, if I added sellers all with the same inventory $I_{s,0}$, as S would grow, the collective sellers position would be $SI_{s,0}$, which would grow as well. This is undesirable because it would mix the effect of increased competition with the effect of an increase in aggregate supply or demand in the underlying asset. Thus I consider some fixed collective initial inventories I_s for sellers, to be split equally across sellers, and I_b for buyers, to be split equally across buyers.

Corollary 1. *As N grows to infinity, holding average market inventory constant, and if the ratio B/S of the number of buyers to the number of sellers remains finite and does not approach zero,*

- *individual traders positions x_k^* in futures contracts shrink to zero,*
- *the total quantity of futures traded (open interest) $B|x_b^*| = S|x_s^*|$ shrinks to zero.*

The second part is not implied by the first: one could imagine that individual traders position shrink to zero just because some fixed quantity of futures is split across more and more traders.

6 The welfare impact of futures

In this section, I derive traders' equilibrium welfare and spell the trade-off between higher trading speed and hedging benefits of futures, observing that trading the futures and the underlying asset in opposite directions implies a welfare loss. Then I show why a higher trading speed benefits to all traders. Finally, I show that overall, futures decrease welfare in this model because traders try to influence futures payoff.

For simplicity, in this section I assume $\mathbb{E}_0[Q] = 0$.

6.1 Traders' welfare with or without futures

Without futures Plugging equilibrium prices and quantities in the certainty equivalent of wealth [3.8](#) leads to

$$\widehat{W}_{k,0}^n = V_k + S_{k,0}(A) + \widehat{S}_{k,1}(A)$$

where $S_{k,0}(A)$ and $\widehat{S}_{k,1}(A)$ are the equilibrium shares of date 0 and date 1 surpluses that accrue to trader k when the rate of trade delay is A :

$$S_{k,0}(A) = \frac{\gamma(\sigma_1^2 + \sigma_2^2)}{2} \left(1 - \left(\frac{A}{1+A} \right)^2 \right) (q_k^c)^2$$

$$\widehat{S}_{k,1}(A) = \frac{\alpha}{2} \gamma \sigma_2^2 z \left(\frac{A}{1+A} q_k^c \right)^2$$

With futures. Similarly, plugging relevant equilibrium quantities into date 0 certainty equivalent of wealth [5.1](#) yields

$$\begin{aligned} \widehat{W}_{k,0}^f &= V_k + S_{k,0}(A_f) + \widehat{S}_{k,1}(A_f) \\ &+ \gamma \tilde{h}_f \left[\left(1 - \frac{\tilde{h}_f}{2} \right) \sigma_1^2 + \left(\alpha - \frac{\tilde{h}_f}{2} \right) \frac{1-z}{\alpha} \sigma_2^2 \right] \left(\frac{A_f}{1+A_f} \right)^2 (q_k^c)^2 \end{aligned} \quad (6.1)$$

where $\tilde{h}_f = \frac{N-2}{N-1} h_f$. It is easy to see that when the hedge ratio h_f is negative, which corresponds to traders trading futures and the underlying asset in opposite directions, the part associated with derivatives becomes negative. Thus the following lemma.

Lemma 2. *If traders choose a negative hedge ratio, i.e. buy futures when they sell the underlying and vice-versa, they make a gross loss.*

Intuitively, as suggested by the discussion of Section [4](#) if traders buy futures when they sell the underlying asset, they increase their exposure to the supply shock Q , and retain too much of the underlying risk on ϵ_1 . However, this goes on with accelerated trading of the underlying asset, which increases welfare as shown in the following subsection.

6.2 Delaying trade decreases traders' welfare

Intuitively, lower quantities traded at the same prices suggests surplus from transactions decreases when more trade is postponed to date 1. The following proposition checks that this is the case, and allows to uncover the different effects that cause the welfare to decrease.

Trading slows down whenever the date 0 rate of trade delay A increases, so that the date-0 quantity decreases and date-1 quantity increases. Thus I measure the effect of trading speed on trader k 's surplus through the marginal effect of an increase in A .

Proposition 8. *Surplus from date 0 and date 1 transactions decrease as the rate A increases, as indicated by the first term of the following:*

$$\frac{\partial(S_{k,0} + \widehat{S}_{k,1})}{\partial A} = -\gamma(\sigma_1^2 + (1 - \alpha z)\sigma_2^2) \frac{A}{(1 + A)^3} (q_{k,0}^c)^2 - (1 - \alpha)\gamma\sigma_2^2 z \mathbb{E}_0 \left[\frac{Q}{N} \right] q_{k,0}^c$$

The second term shows how it affects the distribution of surplus between buyers and sellers.

Welfare decreases for three reasons apparent in the first term

1. Holding cost effect: when $\sigma_1^2 > 0$, sellers hold more risk from date 0 to date 1, which is costlier to them.
2. Uncertainty over the supply shock σ_q^2 makes date 1 surplus is more uncertain, thus decreases its ex ante value; the effect shows up through a lower z .
3. Imperfect competition: given that $\alpha < 1$, risk sharing is ultimately reduced given z .

The second term is non-zero if $\alpha < 1$, which is always the case if N is finite: thus it stems from imperfect competition. It is positive if date 1 customers are expected to sell ($\mathbb{E}_0[Q] > 0$) and trader k is a seller ($q_{k,0}^c < 0$), and vice-versa: traders trading in the opposite direction to date 1 customers are better off postponing trade.

6.3 Net effect of futures introduction

In the equilibrium with futures, trading is faster than without futures, which means a welfare gain. However, traders trade futures in the opposite direction to what they would do if they did it for hedging purposes: this implies a welfare loss. Which force dominates is a priori unclear. The following theorem shows that the negative effect dominates.

Theorem 1. *For all $N \geq 3$, all $\sigma_q^2 \geq 0$ and all $\sigma_1^2 > 0$, introducing futures decrease traders welfare:*

$$\widehat{W}_{k,0}^f < \widehat{W}_{k,0}^n.$$

6.4 The welfare of liquidity traders

The date 1 supply shock Q is a net demand posted by traders whose preferences are not modelled, which in principle precludes computation of their welfare.

However, it is possible to run simple welfare comparisons for them, because their demand is inelastic: modelled traders always absorb all their quantities at a given price. Thus liquidity traders' welfare is measured by the price at which their trades are executed. The date-1 equilibrium price is unaffected by the presence of futures: thus date 1 liquidity traders' welfare is unchanged.

7 A deviation from the law of one price: the futures-spot basis

Comparing spot and futures prices in equations (5.3) and (5.4), one easily sees that there is a non-zero spread ("basis") between the spot and futures price:

$$f_0^* - p_0^* = -\frac{1}{N-2}\gamma\sigma_2^2z\mathbb{E}_0\left[\frac{Q}{N}\right] \quad (7.1)$$

which differs from zero as long as $\mathbb{E}_0[Q] \neq 0$. This basis does *not* stem from the fact that traders seek to influence futures payoff: in the online appendix C I show that with more abstract contracts where payoff influence is impossible the equilibrium prices are the same.

The dependence of the spot price and of the futures prices on $\mathbb{E}_0[Q]$ reflect two different effects and comes in each price with different coefficients. For the spot price, the dependence comes from the certainty equivalent of date-1 surplus $\widehat{S}_{k,1}$, expressed in equation (3.10) and that appears without futures. When traders anticipate sales by customers ($\mathbb{E}_0[Q] > 0$), for instance, they expect to purchase the asset at a low price at $t = 1$. Both to reduce their holding costs and in the hope of realizing a speculative profit by selling at a high price to repurchase at a low price, traders reduce their demands at $t = 0$. The coefficient in front of $\gamma\sigma_2^2z\mathbb{E}_0[Q/N]$ in the marginal wealth of an additional unit of the underlying asset is 1. The effect is symmetric for $\mathbb{E}_0[Q] < 0$. The dependence of the futures price in $\mathbb{E}_0[Q]$ simply reflects the expected payoff from the futures contract, which is $x_k(\widehat{p}_1 - f_0)$ in (5.1). Consistently with the expression of date-1 price, the coefficient of $\gamma\sigma_2^2z\mathbb{E}_0[Q/N]$ is $(N-1)/(N-2) > 1$.

The basis may appear surprising in a context where there are no trading constraints, because traders leave an arbitrage opportunity on the table. This may seem inconsistent with equilibrium:²⁴ when $\mathbb{E}_0[Q] > 0$, the futures is below the spot, so

²⁴The basis is often justified by storage costs (for commodities markets) and interest rate, which

that it is profitable to enter a long position in the futures contract paying off $p_1^* - f_0^*$ and sell the underlying asset giving p_0^* ; at date 1, the futures pays off and the arbitrage strategy would imply re-purchasing the asset at price p_1^* , yielding an overall profit $p_0^* - f_0^*$ for sure. When $\mathbb{E}_0[Q] < 0$, it is profitable to buy the underlying asset and sell the futures.

But such an arbitrage strategy is in fact opposite to traders' optimal strategy. To see it, compare the equilibrium certainty equivalents of wealth for buyers and sellers. The following proposition says that sellers are better off when the futures price increases with respect to the spot price, which means that they are better off by selling the asset at a lower price than the price at which they enter the futures position.

Proposition 9. *Sellers have greater equilibrium utility than buyers if and only if*

$$\frac{S}{N} < u + v \times (f_0^* - p_0^*),$$

where $v > 0$. Otherwise buyers have greater equilibrium utility.

The proof is in the appendix. This proposition therefore indicates if traders were to choose between becoming a buyer or a seller by building inventories or inventory deficits before the date 0 market opens,²⁵ taking $\mathbb{E}_0[Q] \propto f_0^* - p_0^*$ as given, more traders would choose to become sellers of the underlying asset, and buyers of futures, if the futures price f_0^* increased with respect to the spot price p_0^* .

The intuition behind this apparent paradox is the following. When $f_0^* > p_0^*$, the date-0 spot price and the expected date-1 spot price are at a higher level than when $f_0^* < p_0^*$, because $\mathbb{E}_0[Q] < 0$: the terms of trade between buyers and sellers are more favorable to sellers, which raises the welfare associated with selling the underlying asset. In addition, the futures expected payoff $\hat{p}_1 - f_0^*$ does not depend on $\mathbb{E}_0[Q]$, because an increase in the expected date-1 price is reflected one-for-one in the futures price: thus traders' welfare depends on $\mathbb{E}_0[Q]$ only through the date-1 surplus $\hat{S}_{k,1}$, and not through the futures payoff: a higher basis only reflects the impact of a higher spot price on traders' welfare.

The existence of a spread between two assets with identical payoff is usually

are both normalized to zero in the present paper.

²⁵A simple setting would start with all traders having the same inventory position I , with traders observing $\mathbb{E}_0[Q]$, then they would simultaneously choose to build additional inventory for a fixed quantity $\Delta I > 0$ at some cost c .

explained by constraints that preclude traders in one market to trade with traders in another market²⁶, together with constraints on arbitrageurs: valuations for the asset in the two markets differ, so do the prices. In these models, arbitrage allows for the realization of gains from trade and tends to improve market efficiency.²⁷

The present model gives another explanation for the divergence of prices of similar assets, which is clearly not exclusive of constraints. Equilibrium divergence of prices of similar asset should not be surprising in an imperfectly competitive setting: a textbook monopolist prices its good above marginal cost, thus optimally leaves arbitrage opportunities on the table - it is possible to pay the marginal cost to produce a unit of the good, and sell it at a price above the cost. With more competition, the market price for the good would be closer to producers' marginal cost. Here like in traditional models of industrial organization, the spread also goes to zero as the number of traders N increases, *i.e.* as the market becomes more competitive.

Arbitrage opportunities that emerge because of imperfect competition have very different implications than those that emerge because of constraints. While more arbitrage directly benefits traders facing the constraints in the first place, this is not the case here. In the present setting, *arbitrage by traders is a zero-sum game*: if one trader exploits the arbitrage opportunity, traders on the opposite side of both trades make a corresponding loss. It is therefore unlikely that the latter traders would accept such trade. This is in contrast with situations where some traders are constrained not to participate in one market: in these situations, arbitrageurs buy to traders who are eager to sell, and sell to traders who are eager to buy, so that everyone is better off.²⁸ This does not mean that arbitrage would overall decrease welfare: arbitrage for instance could bring more competition, which would presumably increase welfare. But this would go through indirect effects. Full modeling of arbitrageurs would be interesting for future research.

²⁶In Gromb and Vayanos (2002, 2010, 2018), the constraint is exogenous. In Biais et al. (2019), the constraint is of incentive-compatibility.

²⁷In commodities markets, the theory of normal backwardation is often stated as a variant on this theme: in these theories, speculators take the opposite side to traders who need to hedge future production. Speculators require a risk premium for this service. If speculators were allowed to trade in the spot market, they would hedge their futures position with an opposite position in the spot market, so that they would be less exposed to the risk on futures; if they chose perfect hedge, the risk premium would disappear.

²⁸Constraints limiting the arbitrage activity may make arbitrageurs take inefficient positions as shown by Gromb and Vayanos (2002), but if these constraints were not there, arbitrage would unambiguously raise welfare.

8 Conclusion

In this paper, I provide an equilibrium model where futures contracts are traded even if no trader is constrained to trade in the future, and I study pricing and welfare implications. If traders were willing to trade today in the spot market, they would be hedged against future price movements and thus would not need to trade futures. But when traders are imperfectly competitive, traders choose to postpone some trades that would be profitable to execute today in order to minimize their overall price impact. By doing this, they expose themselves to the risk that the at which they will trade tomorrow moves against them. As buyers of the asset fear that the price goes up and sellers fears that the price goes down, there are gains from trading this risk by using futures.

However, once futures are introduced, traders also want to trade the underlying asset at futures maturity in order to influence the futures contract payoff, which is the difference between the underlying asset price at futures maturity, and the futures price: buyers of the futures contract want to raise the underlying asset price, and sellers of the futures want to decrease it. In this setting, they try to impact tomorrow price by trading in the underlying asset.

I show that in equilibrium, this incentive dominates the hedging demand for futures: contrary to what hedging predicts, sellers of the underlying asset *buy* the futures to buyers of the underlying asset, and less trade in the underlying asset is delayed. Sellers of the underlying asset sell more initially in order to sell less tomorrow, which would raise the price. Buyers do the opposite. Moreover, traders attempts to manipulate prices fail in equilibrium, because buyers and sellers pull the price in opposite directions.

Then I assess traders' welfare effects of introducing futures. Attempts to manipulate futures payoff have a positive and a negative effect. The positive effect is that trading in the underlying asset is faster while prices are unchanged, so that sellers carry less of the asset: there is a better allocation of risk associated with the underlying asset. The negative effect is that traders choose negative hedge ratios: trading futures in opposite direction to hedging leaves traders more exposed to tomorrow price risk. Overall, I prove that futures decrease all traders' welfare. However, if manipulation was precluded by offsetting spot price manipulation one-for-one on the futures price, traders would trade futures for hedging, and their welfare would increase.

Finally, I examine how imperfect competition shapes spot and futures prices,

and the basis between the two. This is surprising as bases between two assets with similar payoffs are usually explained with some market segmentation, which is absent in this model.

The futures price can be above or below the spot price, depending on expectations of the supply shock, i.e. depending on expectation of date-1 price. The basis shrinks to zero as competition becomes perfect. I also show that sellers of the underlying asset have greater welfare with respect to sellers when the futures price is more above the spot price, meaning that in equilibrium they sell the underlying asset more cheaply than they buy the futures: this is surprising given that an arbitrageur doing the same would be hurt. While I do not model arbitrageurs, this suggests ambiguous welfare effects of arbitrage in such a context.

Appendix

A Perfect competition benchmark

A.1 No futures

Equilibrium definition. I look for competitive equilibria defined as sets of demand schedules $(q_{i,0}^*(p_0), q_{k,1}^*(p_1))$ ($k = b, s$) and equilibrium prices p_0^c, p_1^c such that:

1. All traders are price-takers;
2. Trader k 's date 1 demand schedule $q_{k,1}^*(p_1)$ maximizes his/her expected utility of terminal wealth W_k given information available at date 1;
3. For each trader k , date 0 demand schedules $q_{k,0}^*(p_0, f_0)$ maximize their expected utility of terminal wealth W_k given information available at date 0 and anticipated equilibrium outcomes at date 1;
4. The market clearing conditions [\(2.2\)](#) and [\(2.4\)](#) hold.

Again I make the slight abuse of notation that that symmetry of traders of class i is included in the definition, while it is in fact an equilibrium outcome. I look for equilibria by backward induction.

A.1.1 Date 1

Traders k maximize over $q_{k,1}$ her expected utility. Given that the only uncertainty is on the normally distributed variable ϵ_2 , the certainty equivalent of wealth is:

$$\widetilde{W}_{k,1} = I_{k,0}v_1 + q_{k,0}(v_1 - p_0) + q_{k,1}(v_1 - p_1) - \frac{\gamma}{2}\sigma_2^2(I_{k,1} + q_{k,1})^2. \quad (\text{A.1})$$

As the utility function is increasing, all happens as if trader k maximized the certainty equivalent $\widetilde{W}_{k,1}$ of her wealth. From the first order condition of this maximization problem one easily derives the optimal competitive demand schedule:

$$q_{k,1}^c(p_1) = \frac{v_1 - p_1}{\gamma\sigma_2^2} - I_{k,1} \quad (\text{A.2})$$

Demand increases when the expected terminal payoff v_1 is larger with respect to the purchase price p_1 , when traders' risk aversion γ is low, and when the terminal payoff

variance σ_2^2 is low. Plugging optimal demands into the market clearing condition (2.4), it is straightforward to derive the equilibrium price

$$p_1^c = v_1 - \gamma\sigma_2^2 \frac{Q_c^*}{2} \quad (\text{A.3})$$

$$\text{with } Q_c^* = \frac{S}{N}I_{s,1} + \frac{B}{N}I_{b,1} + \frac{Q}{N}$$

Notice that with date 0 market clearing condition (2.2), one has $\frac{S}{N}I_{s,1} + \frac{B}{N}I_{b,1} = \frac{S}{N}I_{s,0} + \frac{B}{N}I_{b,0}$. The equilibrium price therefore equals the expected value of the asset minus a risk premium that increases if risk aversions increase, if the uncertainty σ_2^2 over the asset terminal payoff v increases, and if the quantity held by traders after date 1 trade increases. In particular, if customers are net sellers ($Q > 0$), then the equilibrium price decreases and vice versa, which is intuitive.

Plugging equilibrium price (A.3) into optimal demand schedule (A.2), one gets the equilibrium quantities purchased by buyers and sellers are:

$$q_{b,1}^c = \frac{S}{N}(I_{s,1} - I_{b,1}) + \frac{Q}{N}, \quad q_{s,1}^c = \frac{B}{N}(I_{b,1} - I_{s,1}) + \frac{Q}{N} \quad (\text{A.4})$$

After date 1 trade, all traders thus hold

$$I_{b,1} + q_{b,1}^c = I_{s,1} + q_{s,1}^c = \frac{S}{N}I_{s,0} + \frac{B}{N}I_{b,0} + \frac{Q}{N} \quad (\text{A.5})$$

A.1.2 Trader valuation of the surplus of date 1 trade

After date 1 trade, from (A.1), trader k 's certainty equivalent of wealth can be decomposed as

$$\widetilde{W}_{k,1} = I_{k,0}v_1 + q_{k,0}(v_1 - p_0) - \frac{\gamma\sigma_2^2}{2}(I_{k,0} + q_{k,0})^2 + S_{k,1}^c \quad (\text{A.6})$$

$$\text{with } S_{k,1}^c = q_{k,1}^c(v_1 - p_1^c) - \left(\frac{\gamma\sigma_2^2}{2}(I_{k,1} + q_{k,1}^c)^2 - \frac{\gamma\sigma_2^2}{2}(I_{k,1})^2 \right) \quad (\text{A.7})$$

The first terms are the classical mean-variance value of date 0 inventory position after date 0 trade.

S_1^c is the net surplus of date 1 transaction. It is the sum of two terms: $q_{k,1}^c(v_1 - p_1^c)$ is the expected payoff from the trade, while the difference in bracket is the impact of the change in trader k 's inventory position on her risk holding cost. Rearranging

the expression of $S_{k,1}^c$ above leads to:

$$S_{k,1}^c = \frac{\gamma\sigma_2^2}{2} (q_{k,1}^c)^2 \quad (\text{A.8})$$

Crucially, $S_{k,1}^c$ can also be influenced by date 0 trading choice $q_{k,0}$, as revealed by expression [A.4](#) of date-1 quantity. As seen shortly, under perfect competition, this impacts only date-0 equilibrium price. Under imperfect competition, this impacts both date-0 equilibrium price and quantity.

Plugging [A.8](#) into [A.7](#) and taking the certainty equivalent with respect to both ϵ_1 and Q using lemma [5](#) in the appendix,^{[29](#)} one gets the date 0 certainty equivalent of wealth for trader k :

$$\begin{aligned} \widetilde{W}_{k,0} &= I_{k,0}v_0 + q_{k,0}(v_0 - p_0) - \frac{\gamma}{2}(\sigma_1^2 + \sigma_2^2)(I_{k,0} + q_{k,0})^2 \\ &\quad + \frac{\gamma\sigma_2^2}{2} z_c (\mathbb{E}_0[q_{k,1}^c])^2 + cst \end{aligned} \quad (\text{A.9})$$

with $q_{k,1}^c = \frac{Q_c^*}{2} - I_{k,0} - q_{k,0}$ and $z_c = \frac{1}{1 + \gamma^2\sigma_2^2\sigma_q^2}$

A.1.3 Date 0

Optimal demand schedules. The optimal demand $q_{k,0}^c(p_0)$ maximizes the certainty equivalent of wealth [\(A.9\)](#). The problem is solved by the unique solution to the following first order condition:^{[30](#)}

$$v_0 - p_0 = \gamma(\sigma_1^2 + \sigma_2^2) (I_{k,0} + q_{k,0}^c(p_0)) - \gamma\sigma_2^2 z_c [\mathbb{E}_0[Q_c^*] - (I_{k,0} + q_{k,0}^c(p_0))]$$

Rearranging leads to

$$q_{k,0}^c(p_0) = \frac{v_0 - p_0}{\gamma(\sigma_1^2 + (1 - z_c)\sigma_2^2)} - \frac{\sigma_2^2}{\sigma_1^2 + (1 - z_c)\sigma_2^2} z_c \mathbb{E}_0 \left[\frac{Q_c^*}{2} \right] - I_{k,0} \quad (\text{A.10})$$

The optimal demand is the sum of a quasi hold-to-maturity demand (first term), analogous to the two periods demand [\(A.2\)](#), and the short term profit demand, that appears as an arbitrage demand (second term). Both terms are impacted by the uncertainty about the liquidity shock σ_q^2 .

²⁹ $S_{k,1}^c$ is quadratic in Q , and the certainty equivalent takes this into account.

³⁰It is straightforward to check that the problem is strictly concave, as $z_c < 1$.

Equilibrium. The equilibrium prices and quantities are stated in the following proposition, proven in the appendix.

Proposition 10. *The equilibrium price is*

$$p_0^c = v_0 - \gamma(\sigma_1^2 + \sigma_2^2)\bar{I}_0 - \gamma\sigma_2^2 z_c \mathbb{E}_0 \left[\frac{Q}{N} \right] \quad (\text{A.11})$$

where $\bar{I}_0 = \frac{S}{N}I_{s,0} + \frac{B}{N}I_{b,0}$ is the average initial inventory across traders. The risk premium is the sum of a hold-to-maturity component (second term), and of an short-term arbitrage component (third term) that is proportional to the expected date 1 liquidity shock: the price is higher when customer purchases are expected ($\mathbb{E}_0[Q] < 0$) and vice versa. The sensitivity to the date 1 liquidity shock decreases as uncertainty about it increases.

Equilibrium trade and post-trade inventories are

$$q_{b,0}^c = \frac{S}{N}(I_{s,0} - I_{b,0}) \quad q_{s,0}^c = \frac{B}{N}(I_{b,0} - I_{s,0}) \quad (\text{A.12})$$

$$q_{b,1}^c = q_{s,1}^c = \frac{Q}{N} \quad (\text{A.13})$$

and all traders end up with the average inventory $I_0 \equiv S/N I_{s,0} + B/N I_{b,0}$. Risk sharing is Pareto optimal.

Inventories are equalized right after date 0 trade: all intertrader gains from trade are realized at date 0. By market clearing, as date 1 customers are price inelastic, the short term capital gain demand has no impact on the quantities traded and all effect goes in the price.

Traders' welfare. Plugging equilibrium price (A.11) and quantities (A.12) and (A.13) into the certainty equivalent of wealth (A.9) gives

$$\widetilde{W}_{b,0}^c = I_{b,0}v_0 - \frac{\gamma(\sigma_1^2 + \sigma_2^2)}{2}(I_{b,0})^2 + \frac{\gamma(\sigma_1^2 + \sigma_2^2)}{2}(q_{b,0}^c)^2 + \frac{\gamma\sigma_2^2}{2}z_c \left(\mathbb{E}_0 \left[\frac{Q}{N} \right] \right)^2 - \frac{1}{2\gamma} \ln(z_c) \quad (\text{A.14})$$

A.2 Perfect competition: redundancy of futures

Here I add futures contracts, to show that they are perfect substitutes to the underlying asset under competition. To do this I add the futures payoff $x_k(v_1 - p_1^*)$

to the date 1 certainty equivalent of wealth (A.7), rearrange and take the date 0 certainty equivalent of wealth to find

$$\begin{aligned} \widetilde{W}_{k,0}^{c,f} = & I_{k,0}v_0 + q_{k,0}(v_0 - p_0) - \frac{\gamma}{2}(\sigma_1^2 + \sigma_2^2)(I_{k,0} + q_{k,0})^2 + \frac{\gamma\sigma_2^2}{2}z_c(\mathbb{E}_0[q_{k,1}^c])^2 \\ & + x_k(\tilde{p}_1 - f_0) - \frac{\gamma}{2}(\sigma_1^2 + (1 - z_c)\sigma_2^2)(x_k^2 + I_{k,1}x_k) \end{aligned} \quad (\text{A.15})$$

with

$$\tilde{p}_1 = v_0 - \gamma\sigma_2^2z_c\left(\bar{I}_0 + \mathbb{E}_0\left[\frac{Q}{N}\right]\right)$$

and $q_{k,1}^c$ is the date-1 equilibrium quantity (A.13). Differentiating (A.15) with respect to both trade in the underlying asset $q_{k,0}$ and futures x_k , and treating prices including \tilde{p}_0 as constants, I find the following first-order condition:

$$\begin{pmatrix} v_0 - p_0 \\ v_0 - f_0 \end{pmatrix} = \gamma(\sigma_1^2 + (1 - z_c)\sigma_2^2) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} q_{k,0} \\ x_k \end{pmatrix} + cst$$

Clearly, this equation cannot be inverted to get demand schedules. Thus the following proposition.

Proposition 11. *In the competitive equilibrium with futures, quantities traded are indeterminate: futures are redundant with the underlying asset.*

This result illustrates the economic puzzle of futures trading when the underlying asset is also traded. As I shall show in Section 5, under imperfect competition, futures are not redundant anymore.

B Proofs

B.1 Proof of proposition 10

Plugging optimal demand (A.10) into the market clearing condition (2.2), one gets

$$\frac{1}{\gamma} \frac{v_0 - p_0^*}{\sigma_1^2 + \delta_c\sigma_2^2} - \frac{\sigma_2^2z_c}{\sigma_1^2 + \delta_c\sigma_2^2} \mathbb{E}_0[Q_c^*] = 0$$

Rearranging one gets the equilibrium price formula (A.11). Plugging the equilibrium price formula into the optimal demand schedule (A.10), one gets

$$\begin{aligned} I_{k,0} + q_{k,0}^*(p_0^*) &= \frac{\gamma(\sigma_1^2 + \sigma_2^2)\bar{I}_0 + \gamma\sigma_2^2 z_c \mathbb{E}_0 \left[\frac{Q}{N} \right]}{\gamma(\sigma_1^2 + \delta_c \sigma_2^2)} - \frac{\sigma_2^2}{\sigma_1^2 + \delta_c \sigma_2^2} z_c \mathbb{E}_0 [Q_c^*] \\ &= \frac{1}{\sigma_1^2 + \delta_c \sigma_2^2} (\sigma_1^2 + \sigma_2^2 - z_c \sigma_2^2) \bar{I}_0 \\ &= \bar{I}_0 \end{aligned}$$

The optimality of risk sharing comes from the competitiveness of the market (first welfare theorem).

B.2 Proof of proposition 2

B.2.1 Demand schedules

From proposition 1, the post-trade certainty equivalent of wealth at date 1 is given by the following lemma, proven in the appendix.

Lemma 3. *Trader k 's interim expected utility is $-\exp\{-\gamma\widehat{W}_{k,1}\}$, where $\widehat{W}_{k,1}$ is the interim certainty equivalent of wealth given by:*

$$\widehat{W}_{k,1} = I_{k,0}v_1 + q_{k,0}(v_1 - p_0) - \frac{\gamma\sigma_2^2}{2}(I_{k,1})^2 + \alpha \frac{\gamma\sigma_2^2}{2} \left(\frac{\bar{\gamma}}{\gamma} Q^* - I_{k,1} \right)^2 \quad (\text{B.1})$$

$$= I_{k,0}v_1 + q_{k,0}(v_1 - p_0) - \frac{\gamma\sigma_2^2}{2}(I_{k,1})^2 + \frac{N}{N-2} \frac{\gamma\sigma_2^2}{2} (q_{k,1}^*)^2 \quad (\text{B.2})$$

where $\alpha = \frac{N(N-2)}{(N-1)^2} = 1 - \frac{1}{(N-1)^2}$.

Proof. Plugging equilibrium price (3.6) and quantities (3.5) into the date 1 certainty equivalent of wealth (A.1), one gets

$$\widehat{W}_{k,1} = I_{k,0}v_1 + q_{k,0}(v_1 - p_0) - \frac{\gamma\sigma_2^2}{2}(I_{k,1})^2 + q_{k,1}^*(v_1 - p_1^*) - \frac{\gamma\sigma_2^2}{2} \left((I_{k,1} + q_{k,1}^*)^2 - (I_{k,1})^2 \right)$$

Recognizing $\frac{v_1 - p_1^*}{\gamma\sigma_2^2} = \frac{N-1}{N-2}q_{k,1}^* + I_{k,1}$ and rearranging one get

$$\begin{aligned}\widehat{W}_{k,1} &= I_{k,0}v_1 + q_{k,0}(v_1 - p_0) - \frac{\gamma\sigma_2^2}{2}(I_{k,1})^2 \\ &\quad + q_{k,1}^*\gamma\sigma_2^2\left(\frac{N-1}{N-2}q_{k,1}^* + I_{k,1}\right) - \frac{\gamma\sigma_2^2}{2}(2I_{k,1} + q_{k,1}^*)q_{k,1}^* \\ &= I_{k,0}v_1 + q_{k,0}(v_1 - p_0) - \frac{\gamma\sigma_2^2}{2}(I_{k,1})^2 + \left(\frac{N-1}{N-2} - \frac{1}{2}\right)\gamma\sigma_2^2(q_{k,1}^*)^2\end{aligned}$$

which leads to the desired formulas. ■

It is then possible to compute the certainty equivalent of wealth at date 0.

Lemma 4. *The date 0 certainty equivalent of wealth for a buyer is:*

$$\widehat{W}_{b,0} = I_{i,0}v_0 + q_{k,0}(v_0 - p_0) - \frac{\gamma(\sigma_1^2 + \sigma_2^2)}{2}(I_{k,1})^2 + \frac{1}{2}\frac{N}{N-2}\gamma\sigma_2^2z(\mathbb{E}_0[q_{k,1}^*])^2 \quad (\text{B.3})$$

$$\begin{aligned}&= I_{k,0}v_0 + q_{k,0}(v_0 - p_0) - \frac{\gamma(\sigma_1^2 + \sigma_2^2)}{2}(I_{k,1})^2 \\ &\quad + \frac{\alpha}{2}\gamma\sigma_2^2z\left(\frac{\bar{\gamma}}{\gamma}\mathbb{E}_0\left[\frac{Q}{N}\right] + \frac{S}{N}\bar{I}_{s,1}^e + \frac{B-1}{N}\bar{I}_{b,1}^e - \frac{N-1}{N}I_{k,1}\right)^2\end{aligned} \quad (\text{B.4})$$

where $z = (1 + \bar{\gamma}^2\sigma_2^2\sigma_q^2)^{-1}$ and $\bar{\gamma} = \frac{N-1}{N-2}\gamma$. $\bar{I}_{-i,1}^e$ and $\bar{I}_{i,1}^e$ are the expectations of average trader inventories after date 1 trade.

For sellers, replace $\frac{S}{N}\bar{I}_{s,1}^e + \frac{B-1}{N}\bar{I}_{b,1}^e$ with $\frac{S-1}{N}\bar{I}_{s,1}^e + \frac{B}{N}\bar{I}_{b,1}^e$ in (B.4).

Proof. Start from interim expected utility (B.2). Take the certainty equivalent with respect to ϵ_1 first, which gives

$$\widehat{W}_{k,0}|Q = I_{k,0}v_1 + q_{k,0}(v_1 - p_0) - \frac{\gamma(\sigma_1^2 + \sigma_2^2)}{2}(I_{k,1})^2 + \alpha\frac{\gamma\sigma_2^2}{2}\left(\frac{\bar{\gamma}}{\gamma}Q^* - I_{k,1}\right)^2$$

Then take the certainty equivalent with respect to Q is given by the following lemma

Lemma 5. *Let $X \sim \mathcal{N}(\mu, \Sigma)$ a normal vector of dimension p ($|\Sigma| > 0$), and A a symmetric matrix. Then one seeks to compute $\mathbb{E}[\exp(-\gamma X'AX)]$ where A is a symmetric matrix.*

Suppose $I + 2\gamma A\Sigma$ is positive definite, then

$$\mathbb{E}[\exp(-\gamma X'AX)] = \frac{1}{\sqrt{|I + 2\gamma A\Sigma|}} \exp\{-\gamma\mu'(I + 2\gamma A\Sigma)^{-1}A\mu\}$$

Proof.

$$\mathbb{E}[\exp(-\gamma X'AX)] = \int_{\mathbb{R}^p} \frac{1}{\sqrt{2\pi|\Sigma|}} \exp\left\{-\gamma x'Ax - \frac{1}{2}(x - \mu)' \Sigma^{-1}(x - \mu)\right\} dx$$

where $dx \equiv dx_1 dx_2 \dots dx_p$. One first computes

$$\begin{aligned} Q(x) &= -\gamma x'Ax - \frac{1}{2}(x - \mu)' \Sigma^{-1}(x - \mu) \\ &= -\frac{1}{2}(x - \mu)'(\Sigma^{-1} + 2\gamma A)(x - \mu) - 2\gamma \mu' A(x - \mu) - \gamma \mu' A \mu \end{aligned}$$

Suppose that $(\Sigma^{-1} + 2\gamma A)$ is the inverse of a covariance matrix, then the formula will give almost the moment generating function of a normal variable with covariance matrix $[(I + 2\gamma A \Sigma) \Sigma^{-1}]^{-1} = \Sigma(I + 2\gamma A \Sigma)^{-1}$.

$$\begin{aligned} \mathbb{E}\left[e^{-\gamma X'AX}\right] &= \frac{e^{-\gamma \mu' A \mu}}{\sqrt{|I + 2\gamma A \Sigma|}} \\ &\quad \times \int_{\mathbb{R}^p} \frac{1}{\sqrt{2\pi|\Sigma|} |I + 2\gamma A \Sigma|^{-1}} e^{-2\gamma \mu' A(x - \mu)} e^{-\frac{1}{2}(x - \mu)' [\Sigma(I + 2\gamma A \Sigma)^{-1}]^{-1}(x - \mu)} dx \\ &= \frac{1}{\sqrt{|I + 2\gamma A \Sigma|}} \exp\left\{-\gamma \mu' (I + 2\gamma A \Sigma)^{-1} A \mu\right\} \end{aligned}$$

■

For the (B.4), Q^* is an outcome of date 0 trade, as it depends on traders' average inventories in each class. Trader k 's trade has an impact on date his class' average date 1 inventory \bar{I}_1^e since

$$\bar{I}_1^e \equiv \frac{1}{N} \sum_{l=1, l \neq k}^N I_{l,i,1}^e + \frac{I_{k,1}}{N} = \frac{S}{N} I_{s,1}^e + \frac{B-1}{N} I_{b,1}^e + \frac{I_{k,1}}{N}$$

if trader k is a buyer, and

$$\bar{I}_1^e = \frac{S-1}{N} I_{s,1}^e + \frac{B}{N} I_{b,1}^e + \frac{I_{k,1}}{N}$$

if trader k is a seller. I have used the fact that traders within a group (buyers/sellers) play symmetric strategies. ■

Lemma 6. *In equilibrium, all buyers submit the same optimal demand schedules as*

follows:

$$q_{b,0}^*(p_0) = \frac{N-2}{N-1} \left[\frac{v_0 - p_0}{\gamma(\sigma_1^2 + \delta\sigma_2^2)} - I_{b,0} - \frac{N-2}{N-1} z \frac{\sigma_2^2}{\sigma_1^2 + \delta\sigma_2^2} \left(\frac{\bar{\gamma}}{\gamma} \mathbb{E}_0 \left[\frac{Q}{N} \right] + \frac{S}{N} \bar{I}_{s,1}^e + \frac{B-1}{N} \bar{I}_{b,1}^e \right) \right] \quad (\text{B.5})$$

and sellers submit a demand schedule obtained by replacing $\frac{S}{N} \bar{I}_{s,1}^e + \frac{B-1}{N} \bar{I}_{b,1}^e$ with $\frac{S-1}{N} \bar{I}_{s,1}^e + \frac{B}{N} \bar{I}_{b,1}^e$ in (B.5). It depends on trader k 's expectation on other traders' equilibrium trades.

Proof. Differentiate the certainty equivalent of wealth (B.4) with respect to $q_{k,0}$, taking into account its price impact that is conjectured to be constant (and denoted $\lambda_{k,0}$). Equating to zero to get the first-order condition for a buyer:

$$\begin{aligned} v_0 - p_0 &= (\lambda_{k,0} + \gamma(\sigma_1^2 + \sigma_2^2))q_{k,0} + \gamma(\sigma_1^2 + \sigma_2^2)I_{b,0} \\ &\quad + \frac{N-2}{N-1} \gamma \sigma_2^2 z \left(\frac{\bar{\gamma}}{\gamma} \mathbb{E}_0 \left[\frac{Q}{N} \right] + \frac{S}{N} \bar{I}_{s,1}^e + \frac{B-1}{N} \bar{I}_{b,1}^e - \frac{N-1}{N} I_{k,1} \right) \\ &= (\lambda_{k,0} + \gamma(\sigma_1^2 + \delta\sigma_2^2))q_{b,0} + \gamma(\sigma_1^2 + \delta\sigma_2^2)I_{b,0} \\ &\quad + \frac{N-2}{N-1} \gamma \sigma_2^2 z \left(\frac{\bar{\gamma}}{\gamma} \mathbb{E}_0 \left[\frac{Q}{N} \right] + \frac{S}{N} \bar{I}_{s,1}^e + \frac{B-1}{N} \bar{I}_{b,1}^e \right) \end{aligned}$$

where

$$\delta = 1 - \frac{N-2}{N} z \in [0, 1].$$

It is thus straightforward to check that the second derivative of \widehat{W}_{k0} is negative, so that the problem is strictly concave. Using proposition 1 of [Malamud and Rostek \(2017\)](#)

$$\lambda_{k,0} = \frac{\gamma(\sigma_1^2 + \delta\sigma_2^2)}{N-2}$$

Plugging equilibrium price impacts $\lambda_{k,0}$ in the first order condition and rearranging, one gets the desired formula. ■

B.2.2 Equilibrium price and quantities

The date 0 market clearing condition can be written:

$$\frac{v_0 - p_0^*}{\gamma(\sigma_1^2 + \delta\sigma_2^2)} = \bar{I}_0 + \frac{N-2}{N-1} z \frac{\sigma_2^2}{\sigma_1^2 + \delta\sigma_2^2} \left(\frac{\bar{\gamma}}{\gamma} \mathbb{E}_0 \left[\frac{Q}{N} \right] + \frac{N-1}{N} \left(\frac{S}{N} \bar{I}_{s,1}^e + \frac{B}{N} \bar{I}_{b,1}^e \right) \right)$$

By market clearing at date 0, $\frac{S}{N} \bar{I}_{s,1}^e + \frac{B}{N} \bar{I}_{b,1}^e = \frac{S}{N} \bar{I}_{s,0} + \frac{B}{N} \bar{I}_{b,0} \equiv \bar{I}_0$; in addition, recalling $\bar{\gamma}/\gamma = \frac{N-1}{N-2}$, one has

$$v_0 - p_0^* = \gamma(\sigma_1^2 + \delta\sigma_2^2) \bar{I}_0 + \frac{N-2}{N} \gamma \sigma_2^2 z \bar{I}_0 + \gamma \sigma_2^2 z \mathbb{E}_0 \left[\frac{Q}{N} \right]$$

Recalling the definition of δ , the equilibrium price is therefore:

$$p_0^* = v_0 - \gamma(\sigma_1^2 + \sigma_2^2) \bar{I}_0 - \gamma \sigma_2^2 z \mathbb{E}_0 \left[\frac{Q}{N} \right] \quad (\text{B.6})$$

Plugging [B.6](#) into the equilibrium demand schedule for buyers:

$$\begin{aligned} q_{b,0}^* &= \frac{N-2}{N-1} \left[\bar{I}_0 + \frac{N-2}{N-1} z \frac{\sigma_2^2}{\sigma_1^2 + \delta\sigma_2^2} \left(\frac{\bar{\gamma}}{\gamma} \mathbb{E}_0 \left[\frac{Q}{N} \right] + \frac{N-1}{N} \left(\frac{S}{N} \bar{I}_{s,1}^e + \frac{B}{N} \bar{I}_{b,1}^e \right) \right) \right. \\ &\quad \left. - \frac{N-2}{N-1} z \frac{\sigma_2^2}{\sigma_1^2 + \delta\sigma_2^2} \left(\frac{\bar{\gamma}}{\gamma} \mathbb{E}_0 \left[\frac{Q}{N} \right] + \frac{S}{N} \bar{I}_{s,1}^e + \frac{B-1}{N} \bar{I}_{b,1}^e \right) \right] \\ &= \frac{N-2}{N-1} \frac{S}{N} \left[I_{s,0} - I_{b,0} + \frac{1}{N-1} \frac{N-2}{N} z \frac{\sigma_2^2}{\sigma_1^2 + \delta\sigma_2^2} \left(I_{b,0} - I_{s,0} + \left(1 + \frac{B}{S} \right) q_{b,0}^e \right) \right] \end{aligned}$$

where the third line used the equilibrium condition $q_{2,0}^e = q_{2,0}^*$ and market clearing [2.2](#), boiling down to $q_{s,0}^* = -B/S q_{b,0}^*$. Given $1 + B/S = N/S$, one gets:

$$\left(\frac{N-1}{N-2} - \frac{N-2}{N(N-1)} \frac{z\sigma_2^2}{\sigma_1^2 + \delta\sigma_2^2} \right) q_{b,0}^* = \left(1 - \frac{N-2}{N(N-1)} \frac{z\sigma_2^2}{\sigma_1^2 + \delta\sigma_2^2} \right) \frac{S}{N} (I_{s,0} - I_{b,0})$$

Notice that with $\delta = 1 - \frac{N-2}{N} z$,

$$1 - \frac{N-2}{N(N-1)} \frac{z\sigma_2^2}{\sigma_1^2 + \delta\sigma_2^2} = \frac{\sigma_1^2 + \left(1 - \frac{N-2}{N-1} z \right) \sigma_2^2}{\sigma_1^2 + \left(1 - \frac{N-2}{N} z \right) \sigma_2^2} > 0$$

This and rearranging leads to the desired equilibrium quantity:

$$q_{b,0}^* = \frac{1}{1 + A(\sigma_q^2)} \times \underbrace{\frac{S}{N} (I_{s,0} - I_{b,0})}_{= q_{b,0}^e} \quad (\text{B.7})$$

where

$$A(\sigma_q^2) = \frac{1}{N-2} \frac{\sigma_1^2 + \left(1 - \frac{N-1}{N}z\right) \sigma_2^2}{\sigma_1^2 + \left(1 - \frac{N-2}{N-1}z\right) \sigma_2^2}$$

and where the dependence in σ_q^2 in the right-hand side goes through z . This leads to formula [3.14](#). The properties of $A(\sigma_q^2)$ are derived in lemma [7](#) in appendix [B.3](#).

It is also possible to write

$$A(\sigma_q^2) = \underbrace{\frac{1}{N-2}}_{A^{static}} + \frac{\frac{1}{(N-2)(N-1)} z \sigma_2^2}{\underbrace{\sigma_1^2 + \left(1 - \frac{\alpha}{2}z\right) \sigma_2^2}_{A^{dynamic}}}$$

The static rate of trade delay deserves its name because $1/(1 + A^{static}) = \frac{N-2}{N-1}$, which is the same reduction factor as in the date 1 market which is a static game. It is straightforward to show that $A^{dynamic}$ converges to zero as σ_q^2 tends to infinity, so that z converges to 1.

The date-1 quantity is straightforwardly derived from [B.7](#) and [3.5](#).

B.3 Properties of the rate of trade delay $A(\sigma_1^2, \sigma_q^2)$

Define, for $x = \sigma_1^2/\sigma_2^2$ and $z = \frac{1}{1 + \alpha \gamma^2 \sigma_2^2 \sigma_q^2} \in [0, 1]$, the ratio

$$\tilde{A}(x, z) = \frac{1}{N-2} \frac{x + 1 - \frac{N-2}{N}z}{x + 1 - \frac{N-2}{N-1}z}$$

so that $A(\sigma_1^2, \sigma_q^2) = \tilde{A}(x, z)$.

Lemma 7. *Then whatever the finite parameters $N \geq 3$, $\sigma_1^2 \geq 0$ and $\sigma_2^2 > 0$:*

1. $\tilde{A}(x, z)$ is strictly increasing in z so that $A(\sigma_1^2, \sigma_q^2)$ strictly decreases in σ_q^2 .
2. $\tilde{A}(x, z)$ is strictly decreasing in x so that $A(\sigma_1^2, \sigma_q^2)$ strictly decreases in σ_1^2 .
- 3.

$$1 < (N-2)A(\sigma_1^2, \sigma_q^2) < \frac{4}{3}$$

The lower bound 1 is the limit of $(N-2)A(\sigma_1^2, \sigma_q^2)$ when σ_q^2 becomes infinite. The upper bound $4/3$ is attained only in the perfect competition limit ($N \rightarrow \infty$) when both $\sigma_1^2 = 0$ and $\sigma_q^2 = 0$.

4. Therefore

$$\begin{aligned} \frac{1}{N-2} &< A(\sigma_1^2, \sigma_q^2) < \frac{2(N-1)}{N(N-2)} \leq \frac{4}{3} \\ \frac{3}{7} &\leq \frac{N(N-2)}{N^2-2} < \frac{1}{1+A(\sigma_1^2, \sigma_q^2)} < \frac{N-2}{N-1} \\ \frac{1}{N-1} &< \frac{A(\sigma_1^2, \sigma_q^2)}{1+A(\sigma_1^2, \sigma_q^2)} \leq \frac{2N-2}{N^2-2} \leq \frac{4}{7} \end{aligned}$$

Proof. For 1., compute the derivatives

$$\frac{\partial \tilde{A}}{\partial z} = \frac{\sigma_2^2}{N(N-1)} \frac{\sigma_1^2 + \sigma_2^2}{(\sigma_1^2 + (1 - \frac{N-2}{N-1}z)\sigma_2^2)^2} > 0$$

and

$$\frac{\partial \tilde{A}}{\partial x} = -\frac{1}{N(N-1)} \frac{1}{(x+1 - \frac{N-2}{N-1}z)^2} < 0$$

For 2., the first inequality is easily derived from $z \geq 0$; the case $z = 0$ corresponds to $\sigma_q^2 \rightarrow \infty$. For the second inequality, given that $\tilde{A}(\cdot)$ is increasing,

$$(N-2)\tilde{A}(z) \leq (N-2)\tilde{A}(1) = \frac{\frac{\sigma_1^2}{\sigma_2^2} + 1 - \frac{N-2}{N}}{\frac{\sigma_1^2}{\sigma_2^2} + 1 - \frac{N-2}{N-1}} \leq \frac{1 - \frac{N-2}{N}}{1 - \frac{N-2}{N-1}} = 2 \left(1 - \frac{1}{N}\right)$$

where the last inequality follows from the fact that the ratio $\tilde{A}(1)$ is decreasing in the ratio σ_1^2/σ_2^2 . Given that $N \geq 3$, one finally gets the desired inequality

$$(N-2)A(z) \leq \frac{4}{3}.$$

For 3., applying the mappings $x \mapsto 1/(1+x)$ and $x \mapsto x/(1+x)$ to inequalities derived in 2. (all members in these inequalities are greater than -1 so the first mapping reverses ordering, the second preserves it), one gets the desired inequalities. The last inequality is found by applying $N = 2$. \blacksquare

B.4 Equilibrium with futures contracts

The contract has payoff $v_f = p_1^* - f_0$. Trader k purchases a quantity x_k of this contract.

B.4.1 Certainty equivalent of wealth

At date 1 after trade and with the futures payoff, the certainty equivalent of wealth is

$$\begin{aligned}
\widehat{W}_{k,1}^f &= I_{k,0}v_1 + q_{k,0}(v_1 - p_0) - \frac{\gamma\sigma_2^2}{2}(I_{k,1})^2 + \frac{\alpha}{2}\gamma\sigma_2^2 \left(\frac{\bar{\gamma}}{\gamma}Q^* - I_{k,1} \right)^2 \\
&\quad + x_k (v_0 + \epsilon_1 - \bar{\gamma}\sigma_2^2 Q^* - f_0) \\
&= I_{k,0}v_0 + q_{k,0}(v_0 - p_0) + x_k (v_0 - f_0) + (I_{k,1} + x_k)\epsilon_1 \\
&\quad + \frac{\alpha}{2}\gamma\sigma_2^2 \left(\frac{\bar{\gamma}}{\gamma}Q^* - I_{k,1} - \frac{x_k}{\alpha} \right)^2 - \frac{\gamma\sigma_2^2}{2} \left((1 - \alpha)I_{k,1} \right)^2 + \alpha \left(I_{k,1} + \frac{x_k}{\alpha} \right)^2
\end{aligned}$$

Taking the certainty equivalent of wealth with respect to ϵ_1 and Q , one gets

$$\begin{aligned}
\widehat{W}_{k,0}^f &= I_{k,0}v_0 + q_{k,0}(v_0 - p_0) + x_k (v_0 - f_0) + \frac{\alpha}{2}\gamma\sigma_2^2 z \left(\frac{\bar{\gamma}}{\gamma}Q^* - I_{k,1} - \frac{x_k}{\alpha} \right)^2 \\
&\quad - \frac{\gamma}{2} \left(\sigma_1^2(I_{k,1} + x_k) + \sigma_2^2(1 - \alpha)I_{k,1} \right)^2 + \sigma_2^2\alpha \left(I_{k,1} + \frac{x_k}{\alpha} \right)^2 \quad (\text{B.8})
\end{aligned}$$

and developing Q^* for a buyer:

$$\begin{aligned}
\widehat{W}_{b,0}^f &= I_{k,0}v_0 + q_{k,0}(v_0 - p_0) + x_k (v_0 - f_0) \\
&\quad - \frac{\gamma}{2} \left(\sigma_1^2(I_{k,1} + x_k)^2 + \sigma_2^2(1 - \alpha)(I_{k,1})^2 + \sigma_2^2\alpha \left(I_{k,1} + \frac{x_k}{\alpha} \right)^2 \right) \\
&\quad + \frac{\alpha}{2}\gamma\sigma_2^2 z \left(\frac{\bar{\gamma}}{\gamma} \mathbb{E}_0 \left[\frac{Q}{N} \right] + \frac{S}{N}I_{s,1}^e + \frac{B-1}{N}I_{b,1}^e - \frac{N-1}{N}I_{k,1} - \frac{x_k}{\alpha} \right)^2 \quad (\text{B.9})
\end{aligned}$$

Developing and rearranging to separate terms in $I_{k,1}^2$ and terms in x_k^2 leads to expression [5.1](#).

B.4.2 Demand schedules.

Differentiating with respect to $q_{k,0}$ for a buyer:

$$\begin{aligned}
\frac{\partial \widehat{W}_{b,0}^f}{\partial q_{k,0}} &= v_0 - p_0 - \lambda_{qq}^f q_{k,0} - \lambda_{qx}^f x_k - \gamma (\sigma_1^2 (I_{k,1} + x_k) + \alpha \sigma_2^2 I_{k,1} + \alpha \sigma_2^2 (I_{k,1} + x_k/\alpha)) \\
&\quad - \frac{N-2}{N-1} \gamma \sigma_2^2 z \left(\frac{\bar{\gamma}}{\gamma} \mathbb{E}_0 \left[\frac{Q}{N} \right] + \frac{S}{N} I_{s,1}^e + \frac{B-1}{N} I_{b,1}^e - \frac{N-1}{N} I_{k,1} - \frac{x_k}{\alpha} \right) \\
&= v_0 - p_0 - \lambda_{qq}^f q_{k,0} - \lambda_{qx}^f x_k - \gamma (\sigma_1^2 + \delta \sigma_2^2) I_{k,1} - \gamma \left(\sigma_1^2 + \left(1 - \frac{N-1}{N} z \right) \sigma_2^2 \right) x_k \\
&\quad - \frac{N-2}{N-1} \gamma \sigma_2^2 z \left(\frac{\bar{\gamma}}{\gamma} \mathbb{E}_0 \left[\frac{Q}{N} \right] + \frac{S}{N} I_{s,1}^e + \frac{B-1}{N} I_{b,1}^e \right)
\end{aligned}$$

Now differentiating with respect to x_k for a buyer:

$$\begin{aligned}
\frac{\partial \widehat{W}_{b,0}^f}{\partial x_k} &= v_0 - f_0 - \lambda_{xq}^f q_{k,0} - \lambda_{xx}^f x_k - \gamma (\sigma_1^2 (I_{k,1} + x_k) + \sigma_2^2 (I_{k,1} + x_k/\alpha)) \\
&\quad - \gamma \sigma_2^2 z \left(\frac{\bar{\gamma}}{\gamma} \mathbb{E}_0 \left[\frac{Q}{N} \right] + \frac{S}{N} I_{s,1}^e + \frac{B-1}{N} I_{b,1}^e - \frac{N-1}{N} I_{k,1} - \frac{x_k}{\alpha} \right) \\
&= v_0 - f_0 - \lambda_{xq}^f q_{k,0} - \lambda_{xx}^f x_k - \gamma \left(\sigma_1^2 + \left(1 - \frac{N-1}{N} z \right) \sigma_2^2 \right) I_{k,1} - \gamma \left(\sigma_1^2 + \frac{1-z}{\alpha} \sigma_2^2 \right) x_k \\
&\quad - \gamma \sigma_2^2 z \left(\frac{\bar{\gamma}}{\gamma} \mathbb{E}_0 \left[\frac{Q}{N} \right] + \frac{S}{N} I_{s,1}^e + \frac{B-1}{N} I_{b,1}^e \right)
\end{aligned}$$

This leads to the following first order conditions for a buyer, expressed in matrix terms:

$$M_f \begin{pmatrix} v_0 \\ \gamma \sigma_2^2 z \mathbb{E}_0 \left[\frac{Q}{N} \right] \end{pmatrix} - \begin{pmatrix} p_0 \\ f_0 \end{pmatrix} = (\Lambda_f + \gamma \Sigma_f) \begin{pmatrix} q_{k,0}^f(p_0, f_0) \\ x_k^*(p_0, f_0) \end{pmatrix} + \gamma (\Sigma_f + K_f) \begin{pmatrix} I_{k,0} \\ \frac{S}{N} I_{s,1}^e + \frac{B-1}{N} I_{b,1}^e \end{pmatrix} \tag{B.10}$$

where

$$\begin{aligned}
M_f &= \begin{pmatrix} 1 & -1 \\ 1 & -\frac{N-1}{N-2} \end{pmatrix}; & \Lambda_f &= \begin{pmatrix} \lambda_{qq} & \lambda_{qx} \\ \lambda_{xq} & \lambda_{xx} \end{pmatrix} \\
\Sigma_f &= \begin{pmatrix} \sigma_1^2 + \delta \sigma_2^2 & \sigma_1^2 + \left(1 - \frac{N-1}{N} z \right) \sigma_2^2 \\ \sigma_1^2 + \left(1 - \frac{N-1}{N} z \right) \sigma_2^2 & \sigma_1^2 + \frac{1-z}{\alpha} \sigma_2^2 \end{pmatrix} \\
\Sigma_f + K_f &= \begin{pmatrix} \sigma_1^2 + \delta \sigma_2^2 & \frac{N-2}{N-1} z \sigma_2^2 \\ \sigma_1^2 + \left(1 - \frac{N-1}{N} z \right) \sigma_2^2 & z \sigma_2^2 \end{pmatrix}
\end{aligned}$$

Thus

$$K_f = \begin{pmatrix} 0 & -(\sigma_1^2 + (1 - \frac{N-1}{N}(1+\alpha)z)\sigma_2^2) \\ 0 & -\sigma_1^2 + (z - \frac{1-z}{\alpha})\sigma_2^2 \end{pmatrix}$$

B.4.3 Proof of lemma 1

Concavity of $\widehat{W}_{k,0}^f$. For $\widehat{W}_{k,0}^f$ to be strictly concave, I need to show that the first diagonal coefficient of Σ_f is positive, which is easily checked, and that the determinant of Σ_f is positive as well. I compute the determinant of Σ_f :

$$\begin{aligned} |\Sigma_f| &= \left(\sigma_1^2 + \left(1 - \frac{N-2}{N}z\right)\sigma_2^2 \right) \left(\sigma_1^2 + \frac{1-z}{\alpha}\sigma_2^2 \right) - \left(\sigma_1^2 + \left(1 - \frac{N-1}{N}z\right)\sigma_2^2 \right)^2 \\ &= \left(\frac{1}{\alpha} - 1 \right) \sigma_2^2 \left\{ (1-z)\sigma_1^2 + \left(1 - 2\frac{N-1}{N}z\right)\sigma_2^2 \right\} \end{aligned}$$

Given that $\alpha < 1$, the determinant of Σ_f is positive as long as

$$z < \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2 + 2\frac{N-1}{N}\sigma_2^2} \equiv \bar{z}(\sigma_1^2) < 1 \quad (\text{B.11})$$

Thus for low σ_q^2 (i.e. $z > \bar{z}(\sigma_1^2)$), the determinant is *negative* and $\widehat{W}_{k,0}^f$ is not concave, even not quasi-concave.

Trading strategies giving unbounded profit. Suppose σ_q^2 is small enough so that $|\Sigma_f| < 0$; given that the first diagonal coefficient of Σ_f is positive, this implies that the graph of the two-variable mapping $(q_{k,0}, x_k) \mapsto \widehat{W}_{k,0}^f$ is a hyperbolic paraboloid: there are directions $(q_{k,0}, x_k)$ that increase $\widehat{W}_{k,0}^f$, and others that decrease it. As for some constant, a_1 and a_2 ,

$$\widehat{W}_{k,0}^f = cst + a_1 q_{k,0} + a_2 x_k - \underbrace{(q_{k,0}, x_k) \Sigma_f (q_{k,0}, x_k)'}_{Q(q_{k,0}, x_k)}$$

it suffices to exhibit directions for which $(q_{k,0}, x_k) \Sigma_f (q_{k,0}, x_k)' < 0$. To find them, set $q_{k,0} = -ax_k$, where the real number a defines the direction to find. For $x \neq 0$:

$$\frac{Q(-ax, x)}{x^2} = a^2(\sigma_1^2 + \delta\sigma_2^2) - 2a \left(\sigma_1^2 + \left(1 - \frac{N-1}{N}z\right)\sigma_2^2 \right) + \sigma_1^2 + \frac{1-z}{\alpha}\sigma_2^2$$

This polynomial in a has roots (and can be negative for some a) if and only if its discriminant, equal to $-4|\Sigma_f|$, is positive. In this case, the roots are

$$a_{\pm} = \frac{\sigma_1^2 + \left(1 - \frac{N-1}{N}z\right)\sigma_2^2 \pm \sqrt{-|\Sigma_f|}}{\sigma_1^2 + \left(1 - \frac{N-2}{N}z\right)\sigma_2^2}$$

As $-|\Sigma_f|$ remains small, both a_+ and a_- are close to

$$a_0 = \frac{\sigma_1^2 + \left(1 - \frac{N-1}{N}z\right)\sigma_2^2}{\sigma_1^2 + \left(1 - \frac{N-2}{N}z\right)\sigma_2^2} \in (0, 1)$$

Thus it is possible to reach infinite profit by trading infinite quantities provided that $q_{k,0} = -ax_k$, with $a \in (0, 1)$ being such that $Q(-a, 1) < 0$. That is, to trade more futures than the underlying, in opposite directions. QED.

Moreover, trading $q_{k,0} = -a_0x_k$ for each marginal unit of futures x_k achieves the highest marginal value among all a 's.

B.4.4 Demand schedules

Now assume $|\Sigma_f| > 0$, so that $\widehat{W}_{k,0}^f$ is strictly concave. The question is now to determine Λ_f . (Malamud and Rostek [2017]) allow to find it directly:

$$\Lambda_f = \frac{1}{N-2}\gamma\Sigma_f$$

Plugging this into the first order condition, one gets:

$$\begin{aligned} \begin{pmatrix} q_{k,0}^f(p_0, \pi) \\ x_k^*(p_0, f_0) \end{pmatrix} &= \frac{N-2}{N-1}\gamma^{-1}(\Sigma_f)^{-1} \left(M_f \begin{pmatrix} v_0 \\ \gamma\sigma_2^2z\mathbb{E}_0\left[\frac{Q}{N}\right] \end{pmatrix} - \begin{pmatrix} p_0 \\ f_0 \end{pmatrix} \right) \\ &\quad - \frac{N-2}{N-1} (Id_2 + (\Sigma_f)^{-1}K_f) \begin{pmatrix} I_{k,0} \\ \frac{S}{N}\bar{I}_{s,1}^e + \frac{B-1}{N}\bar{I}_{b,1}^e \end{pmatrix} \end{aligned} \quad (\text{B.12})$$

B.4.5 Equilibrium prices

Plugging [B.12] into the market clearing conditions [2.2] and [2.3] allows to find equilibrium risk premia:

$$M_f \begin{pmatrix} v_0 \\ \gamma\sigma_2^2z\mathbb{E}_0\left[\frac{Q}{N}\right] \end{pmatrix} - \begin{pmatrix} p_0^* \\ f_0^* \end{pmatrix} = \gamma(\Sigma_f + K_f) \begin{pmatrix} 1 \\ \frac{N-1}{N} \end{pmatrix} \bar{I}_0 = \begin{pmatrix} \gamma(\sigma_1^2 + \sigma_2^2) \\ \gamma(\sigma_1^2 + \sigma_2^2) \end{pmatrix} \bar{I}_0 \quad (\text{B.13})$$

which leads to

$$p_0^* = v_0 - \gamma(\sigma_1^2 + \sigma_2^2)\bar{I}_0 - \gamma\sigma_2^2 z \mathbb{E}_0 \left[\frac{Q}{N} \right] \quad (\text{B.14})$$

$$f_0^* = v_0 - \gamma(\sigma_1^2 + \sigma_2^2)\bar{I}_0 - \gamma\sigma_2^2 z \frac{N-1}{N-2} \mathbb{E}_0 \left[\frac{Q}{N} \right] \quad (\text{B.15})$$

B.4.6 Equilibrium quantities.

Plugging equilibrium risk premia [B.13](#) into buyers' equilibrium demand schedules [B.12](#):

$$\begin{aligned} \begin{pmatrix} q_{b,0}^f(p_0^f, f_0^*) \\ x_b^*(p_0^f, f_0^*) \end{pmatrix} &= \frac{N-2}{N-1} (Id_2 + \Sigma_f^{-1} K_f) \begin{pmatrix} \bar{I}_0 \\ \frac{N-1}{N} (\frac{S}{N} \bar{I}_{s,1}^e + \frac{B}{N} \bar{I}_{b,1}^e) \end{pmatrix} \\ &\quad - \frac{N-2}{N-1} (Id_2 + \Sigma_f^{-1} K_f) \begin{pmatrix} I_{b,0} \\ \frac{S}{N} \bar{I}_{s,1}^e + \frac{B-1}{N} \bar{I}_{b,1}^e \end{pmatrix} \\ &= \frac{N-2}{N-1} (Id_2 + \Sigma_f^{-1} K_f) \begin{pmatrix} \frac{S}{N} (I_{s,0} - I_{b,0}) \\ \frac{1}{N} \frac{S}{N} (\bar{I}_{b,1}^e - \bar{I}_{s,1}^e) \end{pmatrix} \end{aligned}$$

Denoting κ_1^f and κ_2^f the quantities, to be computed later, such that

$$Id_2 + \Sigma_f^{-1} K_f = \begin{pmatrix} 1 & \kappa_1^f \\ 0 & 1 + \kappa_2^f \end{pmatrix},$$

one gets, using the equilibrium conditions $q_{k,0}^e = q_{k,0}^f(p_0^*, f_0^*)$ and market clearing condition $q_{s,0}^f = -B/S q_{b,0}^f$:

$$\begin{pmatrix} q_{b,0}^f(p_0^*, f_0^*) \\ x_b^*(p_0^*, f_0^*) \end{pmatrix} = \frac{N-2}{N-1} \begin{pmatrix} \frac{S}{N} (I_{s,0} - I_{b,0}) + \frac{\kappa_1^f}{N} \frac{S}{N} (I_{b,0} - I_{s,0} + (1 + \frac{B}{N}) q_{b,0}^f) \\ \frac{1+\kappa_2^f}{N} \frac{S}{N} (I_{b,0} - I_{s,0} + (1 + \frac{B}{N}) q_{b,0}^*) \end{pmatrix}$$

This implies

$$q_{b,0}^f \left(1 - \frac{\kappa_1^f}{N} \frac{N-2}{N-1} \right) = \left(1 - \frac{\kappa_1^f}{N} \right) \frac{N-2}{N-1} \frac{S}{N} (I_{s,0} - I_{b,0})$$

and after rearranging:

$$q_{b,0}^f = \frac{1}{1+A_f} \frac{S}{N} (I_{s,0} - I_{b,0}) \quad (\text{B.16})$$

$$\text{with } A_f = \frac{1}{N-2} \left(1 - \frac{\kappa_1^f}{N}\right)^{-1}$$

Plugging this into the expression for x_b^*

$$x_b^* = \frac{1+\kappa_2^f}{N} \frac{N-2}{N-1} \frac{S}{N} (I_{b,0} + q_{b,0}^e - I_{s,0} - q_{s,0}^e)$$

$$= \frac{1+\kappa_2^f}{N} \frac{N-2}{N-1} \frac{S}{N} \left(I_{b,0} - I_{s,0} + \frac{1}{1+A_f} (I_{s,0} - I_{b,0}) \right).$$

Thus

$$x_b^* = h_f \times \frac{N-2}{N-1} \times \frac{A_f}{1+A_f} \times \frac{S}{N} (I_{s,0} - I_{b,0}) \quad (\text{B.17})$$

$$\text{with } h_f = -\frac{1+\kappa_2^f}{N}$$

Determination of A_f . One has

$$\Sigma_f^{-1} = \frac{1}{|\Sigma_f|} \begin{pmatrix} \sigma_1^2 + \frac{1-z}{\alpha} \sigma_2^2 & -(\sigma_1^2 + (1 - \frac{N-1}{N}z) \sigma_2^2) \\ -(\sigma_1^2 + (1 - \frac{N-1}{N}z) \sigma_2^2) & \sigma_1^2 + \delta \sigma_2^2 \end{pmatrix}$$

so that

$$|\Sigma_f| \kappa_1 = - \left(\sigma_1^2 + \frac{1-z}{\alpha} \sigma_2^2 \right) \left(\sigma_1^2 + \left(1 - \frac{N-1}{N} (1+\alpha)z\right) \sigma_2^2 \right)$$

$$- \left(\sigma_1^2 + \left(1 - \frac{N-1}{N}z\right) \sigma_2^2 \right) \left(-\sigma_1^2 + \left(z - \frac{1-z}{\alpha}\right) \sigma_2^2 \right)$$

$$= -z \sigma_2^2 \left\{ \frac{\sigma_1^2}{N-1} + \frac{\sigma_2^2}{N} \right\}$$

Thus

$$1 - \frac{\kappa_1^f}{N} = 1 + \frac{1}{N} \frac{z \sigma_2^2 \left(\frac{1}{N-1} \sigma_1^2 + \sigma_2^2 / N \right)}{\left(\frac{1}{\alpha} - 1 \right) \sigma_2^2 \left\{ (1-z) \sigma_1^2 + \left(1 - 2 \frac{N-1}{N} z\right) \sigma_2^2 \right\}}$$

$$= 1 + \frac{1}{N} \frac{\alpha}{1-\alpha} \times \frac{z \left(\frac{1}{N-1} \sigma_1^2 + \sigma_2^2 / N \right)}{(1-z) \sigma_1^2 + \left(1 - 2 \frac{N-1}{N} z\right) \sigma_2^2}$$

Recognizing $\frac{1}{N} \frac{\alpha}{1-\alpha} = \frac{1}{N} (N-1)^2 \times \frac{N(N-2)}{(N-1)^2} = N-2$, one gets

$$\begin{aligned} 1 - \frac{\kappa_1^f}{N} &= 1 + \frac{(N-2)z \left(\frac{1}{N-1} \sigma_1^2 + \sigma_2^2/N \right)}{(1-z)\sigma_1^2 + \left(1 - 2\frac{N-1}{N}z\right)\sigma_2^2} \\ &= \frac{\left(1 - \frac{z}{N-1}\right)\sigma_1^2 + (1-z)\sigma_2^2}{(1-z)\sigma_1^2 + \left(1 - 2\frac{N-1}{N}z\right)\sigma_2^2} \end{aligned}$$

Thus

$$A_f = \frac{1}{N-2} \times \frac{(1-z)\frac{\sigma_1^2}{\sigma_2^2} + 1 - 2\frac{N-1}{N}z}{\left[1 - \frac{z}{N-1}\right]\frac{\sigma_1^2}{\sigma_2^2} + 1 - z} \quad (\text{B.18})$$

Determination of κ_2^f . One has

$$\begin{aligned} |\Sigma_f| \kappa_2^f &= \left(\sigma_1^2 + \left(1 - \frac{N-1}{N}z\right)\sigma_2^2 \right) \left(\sigma_1^2 + \left(1 - \frac{N-1}{N}(1+\alpha)z\right)\sigma_2^2 \right) \\ &\quad + \left(\sigma_1^2 + \left(1 - \frac{N-2}{N}z\right)\sigma_2^2 \right) \left(-\sigma_1^2 + \left(z - \frac{1-z}{\alpha}\right)\sigma_2^2 \right) \\ &= \sigma_2^2 \left\{ \sigma_1^2 \left[1 - \frac{1}{\alpha} - \left(\frac{N-2}{N-1} - \frac{1}{\alpha} \right) z \right] \right. \\ &\quad \left. + \sigma_2^2 \left[1 - \frac{1}{\alpha} + \left(1 + \frac{1}{\alpha} + \left(\frac{N-1}{N} \right)^2 - 2\frac{N-1}{N} - \frac{N-2}{N-1} \right) z \right] \right\} \end{aligned}$$

Recognizing $\left(\frac{N-1}{N}\right)^2 - 2\frac{N-1}{N} + 1 = \left(\frac{N-1}{N} - 1\right)^2 = 1/N^2$, I get

$$\begin{aligned} |\Sigma_f| \kappa_2^f &= \sigma_2^2 \left\{ \sigma_1^2 \left[1 - \frac{1}{\alpha} - \left(\frac{N-2}{N-1} - \frac{1}{\alpha} \right) z \right] \right. \\ &\quad \left. + \sigma_2^2 \left[1 - \frac{1}{\alpha} - \left(\frac{N-2}{N-1} - \frac{1}{\alpha} - \frac{1}{N^2} \right) z \right] \right\} \\ &= \left(1 - \frac{1}{\alpha}\right) \sigma_2^2 \left\{ \sigma_1^2 \left[1 - \frac{1/\alpha - \frac{N-2}{N-1}}{1/\alpha - 1} z \right] + \sigma_2^2 \left[1 - \frac{1/\alpha + 1/N^2 - \frac{N-2}{N-1}}{1/\alpha - 1} z \right] \right\} \end{aligned}$$

Therefore

$$\kappa_2^f = - \frac{\left[1 - \frac{1/\alpha - \frac{N-2}{N-1}}{1/\alpha - 1} z \right] \sigma_1^2 + \left[1 - \frac{1/\alpha + 1/N^2 - \frac{N-2}{N-1}}{1/\alpha - 1} z \right] \sigma_2^2}{(1-z)\sigma_1^2 + \left(1 - 2\frac{N-1}{N}z\right)\sigma_2^2}$$

Now the coefficient $h_f = -\frac{1+\kappa_2^f}{N}$ is given by:

$$\begin{aligned} -h_f &= \frac{1}{N} \frac{\left[1 - z - 1 + \frac{1/\alpha - \frac{N-2}{N-1}}{1/\alpha-1} z\right] \sigma_1^2 + \left[1 - 2\frac{N-1}{N}z - 1 + \frac{1/\alpha+1/N^2 - \frac{N-2}{N-1}}{1/\alpha-1} z\right] \sigma_2^2}{(1-z)\sigma_1^2 + \left(1 - 2\frac{N-1}{N}z\right) \sigma_2^2} \\ &= (N-2)z \frac{\frac{1}{N-1}\sigma_1^2 + \left[\frac{(N-1)^2-1}{N(N-2)} - \frac{N-2}{N-1}\right] \sigma_2^2}{(1-z)\sigma_1^2 + \left(1 - 2\frac{N-1}{N}z\right) \sigma_2^2} \end{aligned}$$

Hence

$$h_f = -\frac{N-2}{N-1} \frac{(\sigma_1^2 + \sigma_2^2)z}{(1-z)\sigma_1^2 + \left(1 - 2\frac{N-1}{N}z\right) \sigma_2^2}. \quad (\text{B.19})$$

since $|\Sigma_f| > 0$ to have equilibrium existence, $h_f < 0$: trader k trades the futures at date 0 in the opposite direction as the underlying asset at date 1.

B.4.7 Properties of A_f

A_f decreases with z . Setting $x = \sigma_1^2/\sigma_2^2$ to ease notation:

$$\begin{aligned} \frac{\partial A_f}{\partial z} &= \frac{1}{N-2} \frac{-\left(x + 2\frac{N-1}{N}\right) \left(x + 1 - z\left(\frac{x}{N-1} + 1\right)\right) + \left(\frac{x}{N-1} + 1\right) \left(x + 1 - z\left(x + 2\frac{N-1}{N}\right)\right)}{\left(x + 1 - z\left(\frac{x}{N-1} + 1\right)\right)^2} \\ &= -\frac{1}{N-2} \frac{(x+1)\left(\frac{N-2}{N-1}x + 1 - \frac{2}{N}\right)}{\left(\left(1 - \frac{z}{N-1}\right)x + 1 - z\right)^2} = -\frac{1}{N-1} \frac{(x+1)(x+1-1/N)}{\left(\left(1 - \frac{z}{N-1}\right)x + 1 - z\right)^2} \end{aligned}$$

which is negative as $N \geq 3$ and $x > 0$.

The partial derivative is also decreasing in z , so that A_f is concave in z .

A_f decreases with $x = \sigma_1^2/\sigma_2^2$.

$$\begin{aligned} \frac{\partial A_f}{\partial x} &= \frac{1}{N-2} \frac{(1-z)^2 - \left(1 - 2\frac{N-1}{N}z\right) \left(1 - \frac{z}{N-1}\right)}{\left(\left(1 - \frac{z}{N-1}\right)x + 1 - z\right)^2} = \frac{z}{N-2} \frac{\frac{N-2}{N}z + \frac{1}{N-1} - \frac{2}{N}}{\left(\left(1 - \frac{z}{N-1}\right)x + 1 - z\right)^2} \\ &= \frac{z}{N} \frac{z - \frac{1}{N-1}}{\left(\left(1 - \frac{z}{N-1}\right)x + 1 - z\right)^2} \end{aligned}$$

Thus for $z > \frac{1}{N-1}$, the expression above increases is negative, while for $z < \frac{1}{N-1}$, it is positive. Now A_f is not defined for $z < \bar{z}(\sigma_1^2)$, where $\bar{z}(\sigma_1^2)$ is defined in the proof

of lemma [1](#). It is easy to show that $\bar{z}(\sigma_1^2)$ is minimal for $\sigma_1^2 = 0$, so that

$$\bar{z}(\sigma_1^2) \geq \frac{1}{2} \frac{N}{N-1} \geq \frac{1}{N-1}$$

since $N \geq 3$. Thus $\partial A_f / \partial x < 0$ and A_f decreases with σ_1^2 .

Bounds of A_f . From the variations in z , knowing that $A_f = 0$ for $z = \bar{z}(\sigma_1^2)$ and $A_f = A = \frac{1}{N-2}$ for $z = 0$, one has

$$0 \leq A_f \leq \frac{1}{N-2}$$

B.4.8 Properties of h_f

h_f decreases with z . The numerator of $-h_f$ trivially increases with z , the numerator decreases with z , so $-h_f$ increases with z .

h_f increases with $x = \sigma_1^2 / \sigma_2^2$.

$$\begin{aligned} \frac{\partial h_f}{\partial \sigma_1^2 / \sigma_2^2} &= -\frac{N-2}{N-1} z \frac{(1-z)x + 1 - 2\frac{N-1}{N}z - (x+1)(1-z)}{\left((1-z)x + 1 - 2\frac{N-1}{N}z\right)^2} \\ &= -\frac{N-2}{N-1} \frac{\left(1 - 2\frac{N-1}{N}\right) z^2}{\left((1-z)x + 1 - 2\frac{N-1}{N}z\right)^2} = \frac{N-2}{N-1} \frac{\frac{N-2}{N} z^2}{\left((1-z)x + 1 - 2\frac{N-1}{N}z\right)^2} > 0 \end{aligned}$$

h_f spans the interval $(-\infty, 0]$. It is easy to see from [B.19](#) that h_f decreases with z , thus increases with σ_q^2 . When $z = 0$, $h_f = 0$, and as z converges from above to the value that makes the denominator (proportional to $|\Sigma_f|$ with a positive proportionality constant) approach zero, h_f diverges to $-\infty$.

B.4.9 Proof of proposition [6](#) (manipulable futures)

Here I show that for all σ_1^2, σ_q^2 for which $|\Sigma_f| > 0$,

$$A_f(\sigma_1^2, \sigma_q^2) < A(\sigma_1^2, \sigma_q^2).$$

To do this I examine $A - A_f$ and I proceed in two steps. First I show that for $z = 0$, one has $A = A_f$. this is easily checked by looking at both expressions. Second, I show that $A - A_f$ increases in z for all σ_1^2 and $N \geq 3$, so that $A > A_f$ for $z > 0$.

I know from lemma [7](#) that A increases in z for all σ_1^2 and $N \geq 3$, and that A_f decreases in z for all σ_1^2 and all $N \geq 3$. QED.

B.4.10 Proof of proposition [7](#) and other properties of x_k^*

Proof of proposition [7](#). Trader k 's future position is, using [B.19](#) and [B.18](#):

$$\begin{aligned} x_k^* &= \frac{N-2}{N-1} h_f \frac{A_f}{1+A_f} q_k^c \\ &= -\frac{(N-2)^2}{(N-1)^3} \frac{(\sigma_1^2 + \sigma_2^2)z}{\left(1 - \frac{2N-3}{(N-1)^2}z\right) \sigma_1^2 + \left(1 - \left(\frac{N-2}{N-1} + \frac{2}{N}\right)z\right) \sigma_2^2} q_k^c \end{aligned}$$

As z increases, the numerator of $-x_k^*/q_k^c$ increases, and the denominator decreases, so that $|x_k^*|$ decreases: this proves the first part of the proposition. It is also easy to see that it goes to zero as z goes to zero.

The position x_k^*/q_k^c increases with σ_1^2 . One has

$$\begin{aligned} \frac{\partial}{\partial x} \frac{A_f \tilde{h}_f}{1+A_f} &= \frac{\partial}{\partial x} \left(-\frac{1}{1+A_f} \left(\frac{N-2}{N-1}\right)^2 \frac{(x+1)z}{\left(1 - \frac{z}{N-1}\right) x + 1 - z} \right) \\ &= \frac{-z}{1+A_f} \left(\frac{N-2}{N-1}\right)^2 \\ &\quad \times \left\{ \frac{-\frac{N-2}{N-1}z}{\left(\left(1 - \frac{z}{N-1}\right) x + 1 - z\right)^2} + \frac{x+1}{\left(\left(1 - \frac{z}{N-1}\right) x + 1 - z\right)^2} \frac{1}{1+A_f} \frac{\partial A_f}{\partial x} \right\} \end{aligned}$$

Given that $\frac{\partial A_f}{\partial x} < 0$ (from Section [B.4.7](#)), the term in brackets is negative, so that

$$\frac{\partial}{\partial x} \frac{A_f \tilde{h}_f}{1+A_f} > 0$$

Bounds of x_k^*/q_k^c . I already mentioned that $x_k^* < 0$ and 0 is its limit as σ_q^2 goes to infinity, thus is its upper bound.

I now look for the lower bound. Given that x_k^*/q_k^c decreases with z , it is greater

than the value its expression takes for $z = \bar{z}(\sigma_1^2)$, i.e. given that $A_f = 0$ for $z = \bar{z}$:

$$\begin{aligned} \frac{x_k^*}{q_k^c} &\geq - \left(\frac{N-2}{N-1} \right)^2 \frac{(\sigma_1^2 + \sigma_2^2) \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2 + 2 \frac{N-1}{N} \sigma_2^2}}{\left(1 - \frac{1}{N-1} \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2 + 2 \frac{N-1}{N} \sigma_2^2} \right) \sigma_1^2 + \left(1 - \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2 + 2 \frac{N-1}{N} \sigma_2^2} \right) \sigma_2^2} \\ &= - \frac{N-2}{N-1} \times \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1^2 + \frac{N-1}{N} \sigma_2^2} \equiv \frac{x_k^*(\bar{z}(\sigma_1^2))}{q_k^c} \end{aligned}$$

It is easy to see that $\frac{x_k^*(\bar{z}(\sigma_1^2))}{q_k^c}$ decreases with σ_1^2/σ_2^2 , so that the infimum of the derivative position as a fraction of q_k^c is:

$$\inf_{\sigma_1^2/\sigma_2^2, z} \frac{x_k^*}{q_k^c} = - \frac{N(N-2)}{(N-1)^2} = -\alpha$$

B.4.11 Proof of corollary [1](#) (manipulable futures)

Inspecting the expression h_f , it is clear that as $N \rightarrow \infty$,

$$h_f \rightarrow - \frac{(\sigma_1^2 + \sigma_2^2)z}{(1-z)\sigma_1^2 + (1-2z)\sigma_2^2},$$

which is finite. It is also clear that A_f converges to zero. Finally, given initial inventory positions I_s/S for an individual seller, and I_b/B for an individual buyer, the competitive quantity traded is, from [\(A.12\)](#)

$$q_{b,c} = \frac{I_s - S/BI_b}{N}, \quad q_{b,c} = \frac{I_b - B/SI_s}{N}$$

which shrinks to zero as N grows to infinity and both S/B and B/S remain finite. Therefore, under the condition of the proposition, equilibrium futures trades $x_k^* = \frac{N-2}{N-2} h_f \frac{A_f}{1+A_f} q_k^c$ converge to zero for both buyers and sellers as N diverges.

B.5 Welfare analysis

B.5.1 Welfares with and without futures

Without futures. Plugging equilibrium prices and quantities in the certainty equivalent of wealth [3.8](#) leads to

$$\widehat{W}_{k,0}^n = V_k + S_{k,0}(A) + \widehat{S}_{k,1}(A)$$

where $S_{k,0}(A)$ and $\widehat{S}_{k,1}(A)$ are the equilibrium shares of date 0 and date 1 surpluses that accrue to trader k :

$$S_{k,0}(A) = \frac{\gamma(\sigma_1^2 + \sigma_2^2)}{2} \left(1 - \left(\frac{A}{1+A} \right)^2 \right) (q_k^c)^2 + \gamma\sigma_2^2 z \frac{q_k^c}{1+A} \mathbb{E}_0 \left[\frac{Q}{N} \right]$$

$$\widehat{S}_{k,1}(A) = \frac{\alpha}{2} \gamma \sigma_2^2 z \left(\frac{A}{1+A} \right)^2 (q_k^c)^2 + \gamma \sigma_2^2 z \frac{N}{N-1} \frac{A}{1+A} q_k^c \mathbb{E}_0 \left[\frac{Q}{N} \right] + \frac{\gamma \sigma_2^2 z}{2} \frac{N}{N-2} \left(\mathbb{E}_0 \left[\frac{Q}{N} \right] \right)^2$$

With manipulable futures. Similarly, plugging relevant equilibrium prices and quantities into date 0 certainty equivalent of wealth [5.1](#) yields:

$$\begin{aligned} \widehat{W}_{k,0}^f &= V_k + S_{k,0}(A_f) + \widehat{S}_{k,1}(A_f) \\ &\quad + h_f \frac{N-2}{N-1} \frac{A_f}{1+A_f} q_k^c \left(v_0 - \gamma \sigma_2^2 z \left(\bar{I}_0 + \frac{N-1}{N-2} \mathbb{E}_0 \left[\frac{Q}{N} \right] \right) \right. \\ &\quad \left. - v_0 + \gamma(\sigma_1^2 + \sigma_2^2) \bar{I}_0 + \frac{N-1}{N-2} \mathbb{E}_0 \left[\frac{Q}{N} \right] \right) \\ &\quad - \gamma(\sigma_1^2 + (1-z)\sigma_2^2) \left(I_{k,0} + \frac{q_k^c}{1+A_f} \right) \tilde{h}_f \frac{A_f}{1+A_f} q_k^c \\ &\quad - \frac{\gamma}{2} \left(\sigma_1^2 + \frac{1-z}{\alpha} \sigma_2^2 \right) \tilde{h}_f^2 \left(\frac{A_f}{1+A_f} q_k^c \right)^2 \\ &= V_k + S_{k,0}(A_f) + \widehat{S}_{k,1}(A_f) + \tilde{h}_f \frac{A_f}{1+A_f} q_k^c \gamma(\sigma_1^2 + (1-z)\sigma_2^2) \left(\bar{I}_0 - I_{k,0} - \frac{q_k^c}{1+A_f} \right) \\ &\quad - \frac{\gamma}{2} \left(\sigma_1^2 + \frac{1-z}{\alpha} \sigma_2^2 \right) \tilde{h}_f^2 \left(\frac{A_f}{1+A_f} q_k^c \right)^2 \end{aligned}$$

where to ease notation I denoted $\tilde{h}_f = \frac{N-2}{N-1} h_f$. This leads to

$$\begin{aligned} \widehat{W}_{k,0}^f &= V_k + S_{k,0}(A_f) + \widehat{S}_{k,1}(A_f) \\ &\quad + \gamma \tilde{h}_f \left[\left(1 - \frac{\tilde{h}_f}{2} \right) \sigma_1^2 + \left(\alpha - \frac{\tilde{h}_f}{2} \right) \frac{1-z}{\alpha} \sigma_2^2 \right] \left(\frac{A_f}{1+A_f} \right)^2 (q_k^c)^2 \quad (\text{B.20}) \end{aligned}$$

B.5.2 Proof of proposition 8

I compute

$$\begin{aligned} \frac{\partial S_{k,0} + \widehat{S}_{k,1}}{\partial A} &= -\frac{\gamma(\sigma_1^2 + \sigma_2^2)}{2} (q_{k,0}^c)^2 \times 2 \frac{A}{1+A} \frac{-1}{1+A} - \gamma \sigma_2^2 z q_{k,0}^c \frac{1}{(1+A)^2} \mathbb{E}_0 \left[\frac{Q}{N} \right] \\ &\quad + \frac{N}{N-2} \gamma \sigma_2^2 z \frac{N-2}{N-1} \frac{1}{(1+A)^2} \left(\frac{N-2}{N-1} q_{k,0}^c \frac{A}{1+A} + \mathbb{E}_0 \left[\frac{Q}{N} \right] \right) \\ &= -\gamma(\sigma_1^2 + (1-\alpha z)\sigma_2^2) \frac{A}{(1+A)^3} (q_{k,0}^c)^2 - (1-\alpha)\gamma \sigma_2^2 z \mathbb{E}_0 \left[\frac{Q}{N} \right] q_{k,0}^c \end{aligned}$$

QED.

B.5.3 Proof of theorem 1

Write

$$\begin{aligned} \widehat{W}_{k,0}^f - \widehat{W}_{k,0}^n &= -\gamma \sigma_2^2 (q_k^c)^2 \left(\frac{A_f}{1+A_f} \right)^2 \frac{x+1-\alpha z}{2} \\ &\quad \times \underbrace{\left(1 - \left(\frac{A}{1+A} \frac{1+A_f}{A_f} \right)^2 - 2 \frac{x+1-z}{x+1-\alpha z} \tilde{h}_f + \frac{x+\frac{1-z}{\alpha}}{x+1-\alpha z} \tilde{h}_f^2 \right)}_{\Phi_f(\tilde{h}_f)}. \end{aligned}$$

The goal is to show that $\Phi_f(\tilde{h}_f) > 0$. Now consider the roots of Φ_f : its discriminant is

$$\Delta_{nf} = 4 \frac{x+\frac{1-z}{\alpha}}{x+1-\alpha z} \left[\frac{(x+1-z)^2}{(x+1-\alpha z)(x+\frac{1-z}{\alpha})} - 1 + \left(\frac{A}{1+A} \frac{1+A_f}{A_f} \right)^2 \right]$$

which is always positive because $A > A_f$, so that $\left(\frac{A}{1+A} \frac{1+A_f}{A_f} \right)^2 > 1$. Given that $\tilde{h}_f < 0$, I need to show that \tilde{h}_f is lower than the smallest of the roots of Φ_f , which is

$$h_-^f = \frac{x+1-z}{x+\frac{1-z}{\alpha}} \left\{ 1 - \sqrt{1 + \left(\frac{x+1-\alpha z}{x+1-z} \right)^2 \left(\left(\frac{A}{1+A} \frac{1+A_f}{A_f} \right)^2 - 1 \right)} \right\}$$

Therefore, given the expression of h_-^f and that of \tilde{h}_f , one has $\tilde{h}_f < h_-^f$ if and only if

$$\frac{x + \frac{1-z}{\alpha}}{x+1-z} \frac{A_f \tilde{h}_f}{1+A_f} - \frac{A_f}{1+A_f} < -\sqrt{\left(\frac{A_f}{1+A_f}\right)^2 (1-k^2) + \left(\frac{A}{1+A}\right)^2 k^2} \quad (\text{B.21})$$

where $k = \frac{x+1-\alpha z}{x+1-z}$. In what follows I prove that inequality (B.21) holds in 3 steps. First, I show that the left-hand side increases with z . Second, I show that the right-hand side decreases with z . The first and second step imply that if the inequality holds for the maximum value of z , which is \bar{z} , then it holds for all values of z .

First step. Regarding the left-hand side of (B.21), one has

$$\frac{\partial}{\partial z} \left[\frac{x + \frac{1-z}{\alpha}}{x+1-z} \frac{A_f \tilde{h}_f}{1+A_f} \right] = \frac{\partial}{\partial z} \left[\frac{x + \frac{1-z}{\alpha}}{x+1-z} \right] \frac{A_f \tilde{h}_f}{1+A_f} + \frac{x + \frac{1-z}{\alpha}}{x+1-z} \frac{\partial}{\partial z} \left[\frac{A_f \tilde{h}_f}{1+A_f} \right]$$

It is easy to show that the derivative in the first term is negative, while we know from proposition 5 that $\frac{A_f \tilde{h}_f}{1+A_f}$ is negative, so that the first term is positive. The second term is also positive from the results from Section B.4.7.

Second step. Denote Ψ the term inside the square root of the right-hand side: the RHS decreases with z if and only if Ψ increases with z . One has

$$\frac{\partial \Psi}{\partial z} = \left[\left(\frac{A}{1+A}\right)^2 - \left(\frac{A_f}{1+A_f}\right)^2 \right] \frac{\partial k^2}{\partial z} + \frac{2A_f}{(1+A_f)^3} \frac{\partial A_f}{\partial z} (1-k^2) + \frac{2A}{(1+A)^3} \frac{\partial A}{\partial z} k^2$$

For the first term, it is easy to show that k increases with z , so that $\partial k^2 / \partial z > 0$, while one knows from Proposition 6 that $A_f < A$, which proves that the first factor of the first term is positive. So the first term is positive. For the second term, it is easy to show that $k > 1$, while one has shown in Lemma 7 that $\partial A / \partial z > 0$ and in Section B.4.8 that $\partial A_f / \partial z < 0$. This shows the positivity of $\partial \Psi / \partial z$. QED.

Third step. Given that $\frac{x+1-\alpha z}{x+1-z} > 1$, a sufficient condition for inequality (B.21) to work is

$$\frac{x+1-\alpha z}{x+1-z} \frac{A}{1+A} \frac{1+A_f}{A_f} - \left(\frac{N-2}{N-1}\right)^2 \frac{x + \frac{1-z}{\alpha}}{x+1-z} \frac{(x+1)z}{(1-z)x+1-2\frac{N-1}{N}z} < 1$$

which, after rearranging, leads to

$$\left(\frac{x+1-z}{x+\frac{1-z}{\alpha}} + \left(\frac{N-2}{N-1} \right)^2 \frac{(x+1)z}{(1-z)x+1-2\frac{N-1}{N}z} \right) \frac{A_f}{1+A_f} > \frac{x+1-\alpha z}{x+1-z} \frac{A}{1+A} \quad (\text{B.22})$$

I now prove that this inequality holds for $z = \bar{z}(x)$, where $\bar{z}(x)$ is defined in (B.11). First notice that, as A_f equals zero for $z = \bar{z}$ (but not the product $h_f A_f$, the inequality reduces to

$$\left(\frac{N-2}{N-1} \right)^2 \frac{(x+1)\bar{z}}{(1-\bar{z})x+1-2\frac{N-1}{N}\bar{z}} \frac{A_f(\bar{z})}{1+A_f(\bar{z})} > \frac{x+1-\alpha\bar{z}}{x+1-\bar{z}} \frac{A(\bar{z})}{1+A(\bar{z})}$$

$$\begin{aligned} 1 - \bar{z} &= \frac{N-2}{N} \frac{1}{x+2\frac{N-1}{N}} \\ 1 - \alpha\bar{z} &= \frac{(1-\alpha)x+2\frac{N-1}{N}}{x+2\frac{N-1}{N}} \\ 1 - \frac{\bar{z}}{N-1} &= \frac{\frac{N-2}{N-1}x+2-\frac{2}{N}+\frac{1}{N-1}}{x+2\frac{N-1}{N}} \end{aligned}$$

so that, plugging this into the inequality above leads to

$$\begin{aligned} (x+1)^2 \left(x + \frac{N-1}{N} \right)^2 \\ \geq \left(x^2 + \left(2\frac{N-1}{N} + 1 - \alpha \right) x + 1 - \frac{2}{N} + \frac{1}{(N-1)^2} \right) \left(x + 1 - \frac{2}{3N} \right) \frac{x+1-\frac{N-2}{N(N-1)}}{x+1-\frac{N-2}{N(N-1)^2}} \end{aligned}$$

It is straightforward to show that the left-hand side decreases with N , and not difficult to see that the right-hand side increases with N : the first and second factors are obvious, while the derivative of the third factor with respect to N is

$$\frac{\partial}{\partial N} \left[\frac{x+1-\frac{N-2}{N(N-1)}}{x+1-\frac{N-2}{N(N-1)^2}} \right] = \left(\frac{x+1}{(N-1)^2} - \frac{1}{N^2} \right) \frac{\left(\frac{N-2}{N-1}(x+1) - \frac{1}{N} \right)^4}{\left(\frac{N-1}{N-1}(x+1) + 1 - 1/N \right)^2}$$

which is positive as $x \geq 0$. Thus the above inequality holds for all N if it holds for $N \rightarrow \infty$, i.e., taking the limit, if $(x+1)^4 \geq (x+1)^3$, which holds as long as $x \geq 0$ (and with strict inequality if $x > 0$). Thus for $z = \bar{z}$, $h_-^f \geq \tilde{h}_f$, with strict inequality if $x > 0$. QED.

B.6 Proof of proposition 9

The equilibrium welfare of buyers and sellers is given by (6.1), with

$$S_{k,0}(A) = \frac{\gamma(\sigma_1^2 + \sigma_2^2)}{2} \left(1 - \left(\frac{A}{1+A}\right)^2\right) (q_k^c)^2 + \gamma\sigma_2^2 z \frac{q_k^c}{1+A} \mathbb{E}_0 \left[\frac{Q}{N}\right]$$

$$\widehat{S}_{k,1}(A) = \frac{\alpha}{2} \gamma\sigma_2^2 z \left(\frac{A}{1+A}\right)^2 (q_k^c)^2 + \gamma\sigma_2^2 z \frac{N}{N-1} \frac{A}{1+A} q_k^c \mathbb{E}_0 \left[\frac{Q}{N}\right] + \frac{\gamma\sigma_2^2 z}{2} \frac{N}{N-2} \left(\mathbb{E}_0 \left[\frac{Q}{N}\right]\right)^2$$

For both buyers and sellers, the wealth can be written in the following form:

$$\widehat{W}_{k,0} = I_{k,0}v_0 - \frac{\gamma(\sigma_1^2 + \sigma_2^2)}{2} I_{k,0}^2 + \phi_1 (q_{k,0})^2 + \phi_2 q_{k,0} \mathbb{E}_0 \left[\frac{Q}{N}\right] + \phi_3 \left(\mathbb{E}_0 \left[\frac{Q}{N}\right]\right)^2$$

with

$$\phi_1 = \frac{\gamma(\sigma_1^2 + \sigma_2^2)}{2} + \gamma \left(\frac{A_f}{1+A_f}\right)^2 \left\{ -\frac{\sigma_1^2 + (1-\alpha z)\sigma_2^2}{2} + \tilde{h} \left[\left(1 - \frac{\tilde{h}}{2}\right) \sigma_1^2 + \left(\alpha - \frac{\tilde{h}}{2}\right) \frac{1-z}{\alpha} \sigma_2^2 \right] \right\}$$

$$\phi_2 = \gamma\sigma_2^2 z \frac{1 + \frac{N}{N-1}A_f}{1+A_f}$$

Taking the difference between the expressions for buyers and for sellers, easy algebra gives the expression in the proposition, with

$$u = \frac{1}{2} \left(1 + \frac{v_0 - \gamma(\sigma_1^2 + \sigma_2^2) \frac{I_{s,0} + I_{b,0}}{2}}{\phi_1(I_{s,0} - I_{b,0})}\right), \quad v = \frac{\phi_2}{\phi_1(I_{s,0} - I_{b,0})} \times \frac{N-2}{\gamma\sigma_2^2 z}$$

To show that $v > 0$, given that $\phi_2 > 0$ since $A_f > 0$, I must show that $\phi_1 > 0$. The rest of the proof is dedicated to this.

Proof of $\phi_1 > 0$. Rearranging the second term of ϕ_1 :

$$\frac{2}{\gamma} \phi_1 = \left[1 - \left(\frac{A_f}{1+A_f} (1 - \tilde{h}_f)\right)^2\right] \sigma_1^2$$

$$+ \left[1 - \left(\frac{A_f}{1+A_f} (1 - \tilde{h}_f)\right)^2 - (1-\alpha) \left(z \left(\frac{A_f}{1+A_f}\right)^2 + \frac{1-z}{\alpha} \left(\frac{A_f \tilde{h}_f}{1+A_f}\right)^2\right)\right] \sigma_2^2$$

Now compute, from (B.18) and (B.19), with $x \equiv \sigma_1^2/\sigma_2^2$:

$$\frac{A_f}{1+A_f} \left(1 - \tilde{h}_f\right) = \frac{1}{N-1} \frac{\left(1 - \frac{2N-3}{(N-1)^2} z\right) x + 1 - \left(1 + \frac{N-2}{N(N-1)^2}\right) z}{\left(1 - \frac{2N-3}{(N-1)^2} z\right) x + 1 - \left(1 + \frac{N-2}{N(N-1)^2}\right) z}$$

It is easy to show that the ratio above decreases with x , so is maximized for $x = 0$. Thus

$$\frac{A_f}{1+A_f} \left(1 - \tilde{h}_f\right) \leq \frac{1}{N-1} \frac{1 - \left(1 + \frac{N-2}{N(N-1)^2}\right) z}{1 - \left(1 + \frac{N-2}{N(N-1)^2}\right) z} \leq \frac{1}{(N-1)^2} \equiv 1 - \alpha$$

where the second inequality follows from the fact that the ratio $\frac{1 - \left(1 + \frac{N-2}{N(N-1)^2}\right) z}{1 - \left(1 + \frac{N-2}{N(N-1)^2}\right) z}$ increases with z , so that I take its value at $z = 1$. Therefore

$$\begin{aligned} \frac{2}{\gamma} \phi_1 &\geq [1 - (1 - \alpha)^2] \sigma_1^2 \\ &\quad + \left[1 - (1 - \alpha)^2 - (1 - \alpha) \left(z(1 - \alpha) + (1 - z) \frac{(1 - \alpha)^2}{\alpha}\right)\right] \sigma_2^2 \\ &= [1 - (1 - \alpha)^2] \sigma_1^2 \\ &\quad + \left[1 - (1 - \alpha)^2 \left(1 + z + (1 - z) \frac{1}{\alpha}\right)\right] \sigma_2^2 \\ &\geq [1 - (1 - \alpha)^2] \sigma_1^2 + \left[1 - (1 - \alpha)^2 \left(1 + \frac{1}{\alpha}\right)\right] \sigma_2^2 \\ &= [1 - (1 - \alpha)^2] \sigma_1^2 + \left[1 - (1 - \alpha) \frac{1 - \alpha^2}{\alpha}\right] \sigma_2^2 \end{aligned}$$

The term in σ_1^2 is strictly positive as $\alpha \in (0, 1)$, while the term in σ_2^2 is also positive since $(1 - \alpha) \frac{1 - \alpha^2}{\alpha} = \frac{1 - \alpha^2}{N(N-2)} < 1$. Thus $\phi_1 > 0$.

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C Equilibrium with a theoretical non-manipulable futures contract

Non-manipulable futures contracts. In the present setting, futures payoff manipulation goes through the term $\gamma\sigma_2^2\bar{I}_1^e$ in equation (3.6), where \bar{I}_1^e is the average date 1 inventory when date 1 begins. In order to remove the potential for future payoff manipulation, I assume that any impact on p_1^* is cancelled one-for-one by an opposite effect on the futures price f_0 . That is, I assume that

$$f_0 = -\gamma\sigma_2^2\bar{I}_1^e + \pi, \quad (\text{C.1})$$

where $\bar{I}_1^e = \frac{S}{N}\bar{I}_{s,1}^e + \frac{B}{N}\bar{I}_{b,1}$, so that, given the expression (3.6) of p_1^* , the net payoff of the futures becomes

$$v_y = \epsilon_1 - \bar{\gamma}\sigma_2^2\frac{Q}{N} - \pi \quad (\text{C.2})$$

where again π is to be determined in equilibrium. Although π is not a fee transferred by one party to the other, it acts as “the price” of the contract, so that traders care about the impact of their trades on this price. Trader k purchases a quantity y_k of this contract. Denoting p_0^y the equilibrium price of the underlying asset when this non-manipulable futures are traded, the zero-net supply condition is:

$$\sum_k y_k^*(p_0^y, \pi^*) = 0 \quad (\text{C.3})$$

The certainty equivalent of wealth for trader k is simply found by plugging the assumed form of futures price (C.1) into (5.1). In the appendix I show that it is always concave.

Traders’ wealths. The certainty equivalent of wealth in this case is very similar to (5.1), adapting notations and imposing condition (C.1) on futures price f_0 , as discussed in Section 2.

C.1 Equilibrium prices

Proposition 12. *The equilibrium is unique whatever σ_1^2 and σ_q^2 .*

The underlying asset price p_0^y is equal to the price with imperfect competition and without futures:

$$p_0^y = p_0^n \quad (\text{C.4})$$

The futures price is equal to the price with manipulable futures:

$$f_0^* = v_0 - \gamma(\sigma_1^2 + \sigma_2^2)\bar{I}_0 - \frac{N-1}{N-2}\gamma\sigma_2^2 z \mathbb{E}_0 \left[\frac{Q}{N} \right] \quad (\text{C.5})$$

C.2 Equilibrium trades

Here I show that non-manipulable futures are traded in the opposite direction to ordinary futures, and that they accelerate trading in the underlying asset.

Proposition 13. *In the equilibrium with non-manipulable futures, quantities are:*

$$q_{k,0}^y = \frac{1}{1 + A_y} q_k^c \quad (\text{C.6})$$

$$q_{k,1}^y = \frac{N-2}{N-1} \times \frac{A_y}{1 + A_y} q_k^c + \frac{Q}{N} \quad (\text{C.7})$$

$$y_k^* = h_f \left(q_{k,1}^y - \frac{Q}{N} \right) \quad (\text{C.8})$$

where $A_y > 0$ and $h_y \in (0, 1/2)$.

Proposition 14. *Non-manipulable futures contracts slow down trading in the underlying asset with respect to the case with no contracts, while manipulable futures accelerate trading:*

$$\begin{aligned} |q_{k,0}^y| &< |q_{k,0}^*| < |q_{k,0}^f| \\ |q_{k,1}^y| &> |q_{k,1}^*| > |q_{k,1}^f| \end{aligned}$$

This is because the rates of trade delay are such that $A_y > A > A_f$.

Non-manipulable futures are a portfolio of a contract that allows to share risk over the supply shock Q , whose need was stated in Section 4.1, and of a contract that allows to share risk about information ϵ_1 (subsection 4.2). The ϵ_1 component of non-manipulable futures allows to share the costs associated with sellers holding the underlying asset from date 0 to date 1: this gives sellers extra incentives to carry more of the underlying asset until date 1. Similarly, sharing the risk over Q

allows traders to make the transaction costs benefit of postponing trade from date 0 to date 1 not to be offset by the cost of a high uncertainty over the date-1 price level. Proposition ?? thus suggests that a hedging of this risk would slow the down trading pace. Thus sharing risk over ϵ_1 and Q allows by different mechanisms sellers to carry inventory for longer.

C.3 Welfare

Non-manipulable futures slow down trading in the asset, which entails welfare cost as shown above, but bring hedging benefit. The following theorem shows that the latter dominates.

Theorem 2. *Introducing non-manipulable futures raises traders' welfare:*

$$\widehat{W}_{k,0}^y > \widehat{W}_{k,0}^n$$

C.4 Resolution of date-0 equilibrium with non-manipulable futures contracts

I study the contract with gross payoff

$$v_y = \epsilon_1 - \bar{\gamma}\sigma_2^2 \frac{Q}{N}$$

C.4.1 Certainty equivalent of wealth

I first compute the date 0 certainty equivalent of wealth as a function of $q_{k,0}$ and y_k . After date 1 trade, the certainty equivalent of wealth of trader k is, from [3.14](#) and [3.6](#) and the derivative payoff

$$\begin{aligned} \widehat{W}_{k,1}^y &= I_{k,0}v_0 + q_{k,0}(v_0 - p_0) - y_k\pi + (I_{k,1} + y_k)\epsilon_1 - \frac{\gamma\sigma_2^2}{2}(I_{k,1})^2 \\ &\quad + \frac{\alpha\gamma\sigma_2^2}{2} \left(\frac{\bar{\gamma}}{\gamma} \mathbb{E}_0 [Q^*] - I_{k,1} \right)^2 - y_k\bar{\gamma}\sigma_2^2 \frac{Q}{N} \\ &= I_{k,0}v_0 + q_{k,0}(v_0 - p_0) - b_k\pi_b + (I_{k,1} + y_k)\epsilon_1 - \frac{\gamma}{2} \left((1 - \alpha)\sigma_2^2(I_{k,1})^2 + \alpha\sigma_2^2 \left(I_{k,1} + \frac{y_k}{\alpha} \right)^2 \right) \\ &\quad + \frac{\alpha\gamma\sigma_2^2}{2} \left(\frac{\bar{\gamma}}{\gamma} Q^* - \left(I_{k,1} + \frac{y_k}{\alpha} \right) \right)^2 + \gamma\sigma_2^2 \left(\frac{S}{N} \bar{I}_{s,1}^e + \frac{B}{N} \bar{I}_{b,1}^e \right) y_k \end{aligned}$$

Taking the certainty equivalent of wealth with respect to ϵ_1 and Q leads to

$$\begin{aligned}\widehat{W}_{k,0}^y &= I_{k,0}v_0 + q_{k,0}(v_0 - p_0) - y_k\pi - \frac{\gamma}{2} \left(\sigma_1^2(I_{k,1} + y_k)^2 + \alpha\sigma_2^2 \left(I_{k,1} + \frac{y_k}{\alpha} \right)^2 + (1 - \alpha)(I_{k,1})^2 \right) \\ &\quad + \frac{\alpha}{2}\gamma\sigma_2^2z \left(\frac{\bar{\gamma}}{\gamma} \mathbb{E}_0 [Q^*] - I_{k,1} - \frac{y_k}{\alpha} \right)^2 + \gamma\sigma_2^2 \left(\frac{S}{N}\bar{I}_{s,1}^e + \frac{B}{N}\bar{I}_{b,1}^e \right) y_k\end{aligned}$$

Taking the expression in [3.6](#) for Q^* leads to the desired formula.

C.4.2 Demand schedules

Lemma 8. *For non-manipulable futures, the marginal valuation of the underlying asset and the futures are, for a buyer:*

$$\begin{aligned}\frac{\partial \widehat{W}_{b,0}^y}{\partial q_{k,0}} &= v_0 - p_0 - \lambda_{qq}^y q_{k,0} - \lambda_{qb}^y b_k - \gamma(\sigma_1^2 + \delta\sigma_2^2)I_{k,1} - \gamma \left(\sigma_1^2 + \frac{N-1}{N}(1-z)\sigma_2^2 \right) y_k \\ &\quad - \gamma\sigma_2^2z \mathbb{E}_0 \left[\frac{Q}{N} \right] - \frac{N-2}{N-1}\gamma\sigma_2^2z \left(\frac{S}{N}\bar{I}_{s,1}^e + \frac{B-1}{N}\bar{I}_{b,1}^e \right)\end{aligned}$$

with $\delta = 1 - \frac{N-1}{N}z$ and

$$\begin{aligned}\frac{\partial \widehat{W}_{b,0}^y}{\partial y_k} &= -\pi - \lambda_{yq}^y q_{k,0} - \lambda_{yy}^y y_k - \gamma \left(\sigma_1^2 + \frac{N-1}{N}(1-z)\sigma_2^2 \right) I_{k,1} - \gamma \left(\sigma_1^2 + \frac{1-z}{\alpha}\sigma_2^2 \right) y_k \\ &\quad - \frac{N-1}{N-2}\gamma\sigma_2^2z \mathbb{E}_0 \left[\frac{Q}{N} \right] + \gamma\sigma_2^2(1-z) \left(\frac{S}{N}\bar{I}_{s,1}^e + \frac{B-1}{N}\bar{I}_{b,1}^e \right)\end{aligned}$$

Then differentiating (??) with respect to $q_{k,0}$ for a buyer:

$$\begin{aligned}\frac{\partial \widehat{W}_{b,0}^y}{\partial q_{k,0}} &= v_0 - p_0 - \lambda_{qq}^y q_{k,0} - \lambda_{qy}^y y_k - \gamma \left(\sigma_1^2(I_{k,1} + y_k) + \alpha\sigma_2^2 \left(I_{k,1} + \frac{y_k}{\alpha} \right) + (1 - \alpha)\sigma_2^2 I_{k,1} \right) \\ &\quad - \frac{N-1}{N}\alpha\gamma\sigma_2^2z \left(\frac{\bar{\gamma}}{\gamma} \mathbb{E}_0 \left[\frac{Q}{N} \right] + \frac{S}{N}\bar{I}_{s,1}^e + \frac{B-1}{N}\bar{I}_{b,1}^e - \frac{N-1}{N}I_{k,1} - \frac{y_k}{\alpha} \right) \\ &\quad + \frac{1}{N}\gamma\sigma_2^2y_k \\ &= v_0 - p_0 - \lambda_{qq}^y q_{k,0} - \lambda_{qb}^y b_k - \gamma(\sigma_1^2 + \sigma_2^2)I_{k,1} + \left(\sigma_1^2 + \frac{N-1}{N}\sigma_2^2 \right) y_k \\ &\quad - \frac{N-2}{N-1}\gamma\sigma_2^2z \left(\frac{\bar{\gamma}}{\gamma} \mathbb{E}_0 \left[\frac{Q}{N} \right] + \frac{S}{N}\bar{I}_{s,1}^e + \frac{B-1}{N}\bar{I}_{b,1}^e - \frac{N-1}{N}I_{k,1} - \frac{y_k}{\alpha} \right)\end{aligned}$$

Rearranging leads to

$$\begin{aligned} \frac{\partial \widehat{W}_{b,0}^y}{\partial q_{k,0}} &= v_0 - p_0 - \lambda_{qq}^y q_{k,0} - \lambda_{qb}^y b_k - \gamma(\sigma_1^2 + \delta\sigma_2^2)I_{k,1} - \gamma\left(\sigma_1^2 + \frac{N-1}{N}(1-z)\sigma_2^2\right)y_k \\ &\quad - \gamma\sigma_2^2 z \mathbb{E}_0\left[\frac{Q}{N}\right] - \frac{N-2}{N-1}\gamma\sigma_2^2 z \left(\frac{S}{N}\bar{I}_{s,1}^e + \frac{B-1}{N}\bar{I}_{b,1}^e\right) \end{aligned} \quad (\text{C.9})$$

with $\delta = 1 - \frac{N-1}{N}z$.

Then differentiating ?? with respect to y_k :

$$\begin{aligned} \frac{\partial \widehat{W}_{b,0}^y}{\partial y_k} &= -\pi - \lambda_{yq}^y q_{k,0} - \lambda_{yy}^y y_k - \gamma\left(\sigma_1^2(I_{k,1} + y_k) + \sigma_2^2\left(I_{k,1} + \frac{y_k}{\alpha}\right)\right) \\ &\quad - \gamma\sigma_2^2 z \left(\frac{\bar{\gamma}}{\gamma} \mathbb{E}_0\left[\frac{Q}{N}\right] + \frac{S}{N}\bar{I}_{s,1}^e + \frac{B-1}{N}\bar{I}_{b,1}^e - \frac{N-1}{N}I_{k,1} - \frac{y_k}{\alpha}\right) \\ &\quad + \gamma\sigma_2^2 \left(\frac{S}{N}\bar{I}_{s,1}^e + \frac{B}{N}\bar{I}_{b,1}^e\right) \\ &= -\pi - \lambda_{yq}^y q_{k,0} - \lambda_{yy}^y y_k - \gamma(\sigma_1^2 + \sigma_2^2)I_{k,1} - \gamma\left(\sigma_1^2 + \frac{\sigma_2^2}{\alpha}\right)y_k \\ &\quad - \gamma\sigma_2^2 z \left(\frac{\bar{\gamma}}{\gamma} \mathbb{E}_0\left[\frac{Q}{N}\right] + \frac{S}{N}\bar{I}_{s,1}^e + \frac{B-1}{N}\bar{I}_{b,1}^e - \frac{N-1}{N}I_{k,1} - \frac{y_k}{\alpha}\right) \\ &\quad + \gamma\sigma_2^2 \left(\frac{S}{N}\bar{I}_{s,1}^e + \frac{B-1}{N}\bar{I}_{b,1}^e + \frac{I_{k,1}}{N}\right) \end{aligned}$$

Rearranging leads to

$$\begin{aligned} \frac{\partial \widehat{W}_{b,0}^y}{\partial y_k} &= -\pi - \lambda_{yq}^y q_{k,0} - \lambda_{yy}^y y_k - \gamma\left(\sigma_1^2 + \frac{N-1}{N}(1-z)\sigma_2^2\right)I_{k,1} - \gamma\left(\sigma_1^2 + \frac{1-z}{\alpha}\sigma_2^2\right)y_k \\ &\quad - \frac{N-1}{N-2}\gamma\sigma_2^2 z \mathbb{E}_0\left[\frac{Q}{N}\right] + \gamma\sigma_2^2(1-z)\left(\frac{S}{N}\bar{I}_{s,1}^e + \frac{B-1}{N}\bar{I}_{b,1}^e\right) \end{aligned} \quad (\text{C.10})$$

which proves lemma [8](#).

C.4.3 Proof of proposition [4](#) (non-manipulable futures)

Optimal demand schedules. From [C.9](#) and [C.10](#), the first order conditions can be expressed in matrix form as follows:

$$M_y \begin{pmatrix} v_0 \\ \gamma\sigma_2^2 z \mathbb{E}_0\left[\frac{Q}{N}\right] \end{pmatrix} - \begin{pmatrix} p_0 \\ \pi \end{pmatrix} = (\Lambda_y + \gamma\Sigma_y) \begin{pmatrix} q_{k,0} \\ y_k \end{pmatrix} + \gamma(\Sigma_y + K_y) \begin{pmatrix} I_{b,0} \\ \frac{S}{N}\bar{I}_{s,1}^e + \frac{B-1}{N}\bar{I}_{b,1}^e \end{pmatrix} \quad (\text{C.11})$$

where

$$\begin{aligned} M_y &= \begin{pmatrix} 1 & -1 \\ 0 & -\frac{N-1}{N-2} \end{pmatrix}, \\ \Sigma_y &= \begin{pmatrix} \sigma_1^2 + \delta\sigma_2^2 & \sigma_1^2 + \frac{N-1}{N}(1-z)\sigma_2^2 \\ \sigma_1^2 + \frac{N-1}{N}(1-z)\sigma_2^2 & \sigma_1^2 + \frac{1-z}{\alpha}\sigma_2^2 \end{pmatrix}, \\ \Sigma_y + K_y &= \begin{pmatrix} \sigma_1^2 + \delta\sigma_2^2 & \frac{N-2}{N-1}z\sigma_2^2 \\ \sigma_1^2 + \frac{N-1}{N}(1-z)\sigma_2^2 & -(1-z)\sigma_2^2 \end{pmatrix} \end{aligned}$$

so that

$$K_y = \begin{pmatrix} 0 & -(\sigma_1^2 + \frac{N-1}{N}(1-(1+\alpha)z)\sigma_2^2) \\ 0 & -(\sigma_1^2 + (\frac{1}{\alpha} + 1)(1-z)\sigma_2^2) \end{pmatrix}$$

Now I compute $(\Sigma_y)^{-1}$. The determinant of Σ_y is

$$\begin{aligned} |\Sigma_y| &= \left(\sigma_1^2 + \left(1 - \frac{N-2}{N}z\right)\sigma_2^2 \right) \left(\sigma_1^2 + \frac{1-z}{\alpha}\sigma_2^2 \right) - \left(\sigma_1^2 + \frac{N-1}{N}(1-z)\sigma_2^2 \right)^2 \\ &= \sigma_2^2 \left\{ \left[\frac{2}{N} + \left(\frac{1}{\alpha} - 1\right)(1-z) \right] \sigma_1^2 + \frac{2}{N} \frac{1-z}{\alpha} \sigma_2^2 \right\} \end{aligned}$$

Thus

$$|\Sigma_y| = \frac{2}{N}\sigma_2^2 \left\{ \left[1 + \frac{1-z}{2(N-2)} \right] \sigma_1^2 + \frac{1-z}{\alpha}\sigma_2^2 \right\} \quad (\text{C.12})$$

which is always positive since $z \leq 1$: trader k 's problem is strictly concave. And

$$(\Sigma_y)^{-1} = |\Sigma_y|^{-1} \begin{pmatrix} \sigma_1^2 + \frac{1-z}{\alpha}\sigma_2^2 & -(\sigma_1^2 + \frac{N-1}{N}(1-z)\sigma_2^2) \\ -(\sigma_1^2 + \frac{N-1}{N}(1-z)\sigma_2^2) & \sigma_1^2 + \delta\sigma_2^2 \end{pmatrix}$$

Equilibrium prices. Similarly to the case with manipulable futures, equilibrium risk premia are:

$$M_y \begin{pmatrix} v_0 \\ \gamma\sigma_2^2 z \mathbb{E}_0 \left[\frac{Q}{N} \right] \end{pmatrix} - \begin{pmatrix} p_0^y \\ \pi^* \end{pmatrix} = \gamma(\Sigma_y + K_y) \begin{pmatrix} 1 \\ \frac{N-1}{N} \end{pmatrix} \bar{I}_0 = \begin{pmatrix} \gamma(\sigma_1^2 + \sigma_2^2) \\ \gamma\sigma_1^2 \end{pmatrix} \bar{I}_0$$

This leads to

$$\begin{aligned} p_0^y &= v_0 - \gamma(\sigma_1^2 + \sigma_2^2)\bar{I}_0 - \gamma\sigma_2^2 z \mathbb{E}_0 \left[\frac{Q}{N} \right] \\ \pi^* &= -\gamma\sigma_1^2 \bar{I}_0 - \frac{N-1}{N-2} \gamma\sigma_2^2 z \mathbb{E}_0 \left[\frac{Q}{N} \right] \end{aligned}$$

C.4.4 Proof of proposition [5](#)

Similarly to the case with manipulable futures:

$$\begin{pmatrix} q_{b,0}^b(p_0^y, \pi^*) \\ y_b^*(p_0^*, \pi^*) \end{pmatrix} = \frac{N-2}{N-1} (Id_2 + \Sigma_y^{-1} K_y) \begin{pmatrix} \frac{S}{N} (I_{s,0} - I_{b,0}) \\ \frac{1}{N} \frac{S}{N} (\bar{I}_{b,1}^e - \bar{I}_{s,1}^e) \end{pmatrix}$$

Denoting κ_1^y and κ_2^y the quantities, to be computed later, such that

$$Id_2 + \Sigma_y^{-1} K_y = \begin{pmatrix} 1 & \kappa_1^y \\ 0 & 1 + \kappa_2^y \end{pmatrix},$$

one gets, similarly to the case with manipulable futures:

$$\begin{aligned} q_{b,0}^y &= \frac{1}{1 + A_y} \frac{S}{N} (I_{s,0} - I_{b,0}) & (C.13) \\ \text{with } A_y &= \frac{1}{N-2} \left(1 - \frac{\kappa_1^y}{N} \right)^{-1} \end{aligned}$$

Plugging this into the expression for y_b^* leads to

$$y_b^* = -\frac{N-2}{N-1} \times \frac{1 + \kappa_2^y}{N} \times \frac{A_y}{1 + A_y} \times \frac{S}{N} (I_{s,0} - I_{b,0}) \quad (C.14)$$

Computation of κ_1^y and $q_{b,0}^y$. One has

$$\Sigma_y^{-1} K_y = \begin{pmatrix} 0 & \kappa_1^y \\ 0 & \kappa_2^y \end{pmatrix}$$

where

$$\begin{aligned}
|\Sigma_y| \kappa_1^y &= - \left(\sigma_1^2 + \frac{1-z}{\alpha} \sigma_2^2 \right) \left(\sigma_1^2 + \frac{N-1}{N} (1 - (1+\alpha)z) \right) \\
&\quad + \left(\sigma_1^2 + \frac{N-1}{N} (1-z) \sigma_2^2 \right) \left(\sigma_1^2 + \left(1 + \frac{1}{\alpha}\right) (1-z) \sigma_2^2 \right) \\
&= \sigma_2^2 \left\{ \left(1 - \frac{z}{N-1}\right) \sigma_1^2 + \frac{N-1}{N} (1-z) \sigma_2^2 \right\}
\end{aligned}$$

Therefore

$$\begin{aligned}
1 - \frac{\kappa_1^y}{N} &= 1 - \frac{\left(1 - \frac{z}{N-1}\right) \sigma_1^2 + \frac{N-1}{N} (1-z) \sigma_2^2}{\left[2 + \frac{1-z}{N-2}\right] \sigma_1^2 + \frac{2}{\alpha} (1-z) \sigma_2^2} \\
&= \frac{N-1}{N-2} \frac{\left(1 - \frac{z}{(N-1)^2}\right) \sigma_1^2 + (1-z) \sigma_2^2}{\left[2 + \frac{1-z}{N-2}\right] \sigma_1^2 + \frac{2}{\alpha} (1-z) \sigma_2^2}
\end{aligned}$$

so that

$$A_y(\sigma_q^2) = \frac{1}{N-1} \times \frac{\left(2 + \frac{1-z}{N-2}\right) \sigma_1^2 + \frac{2}{\alpha} (1-z) \sigma_2^2}{\left(1 - \frac{z}{(N-1)^2}\right) \sigma_1^2 + (1-z) \sigma_2^2} \quad (\text{C.15})$$

Together with [C.13](#), this gives equilibrium date-0 quantity traded $q_{b,0}^y$.

When simultaneously $\sigma_1^2 = 0$ and $z = 1$, A_y may appear undefined; it is easy to see that when σ_1^2 converges to zero and z converges to 1, whatever the order in taking the limits, A_y converges to $2 \frac{N-1}{N(N-2)}$. So I define this value for A_y when $\sigma_1^2 = 0$ and $z = 1$.

Variations of A_y Denote $x = \sigma_1^2 / \sigma_2^2$. Then

$$\begin{aligned}
\frac{\partial A_y}{\partial x} &= \frac{2}{N-1} \frac{\left(1 + \frac{1-z}{2(N-2)}\right) \left(\left(1 - \frac{z}{(N-1)^2}\right) x + (1-z)\right) - \left(1 - \frac{z}{(N-1)^2}\right) \left(\left(1 + \frac{1-z}{2(N-2)}\right) x + \frac{1-z}{\alpha}\right)}{(\dots)^2} \\
&= \frac{2}{N-1} \frac{\left(\frac{1}{2} - \frac{1}{N}\right) (1-z) \left(\frac{1}{N-2} + z\right)}{\left(\left(1 - \frac{z}{(N-1)^2}\right) \sigma_1^2 + (1-z) \sigma_2^2\right)^2} \\
&> 0
\end{aligned}$$

given that $N \geq 3$, so that $1/2 > 1/N$. Differentiating with respect to z :

$$\begin{aligned} \frac{\partial A_y}{\partial z} &= \frac{2}{N-1} \frac{-\left(\frac{1}{\alpha} + \frac{x}{2(N-2)}\right) \left((1 - (1-\alpha)z)x + 1 - z\right) + \left((1-\alpha)x + 1\right) \left(\left(1 + \frac{1-z}{2(N-2)}\right)x + \frac{1-z}{\alpha}\right)}{(\dots)^2} \\ &= \frac{2}{N-1} \frac{x^2}{(\dots)^2} \left\{ 1 - \alpha - \frac{\alpha}{2(N-2)} \right\} \\ &= \frac{2}{N-1} \frac{x^2}{(\dots)^2} \left\{ 1 - \frac{N(N-3/2)}{(N-1)^2} \right\} \end{aligned}$$

Given that for all N , $\frac{N(N-3/2)}{(N-1)^2} > 1$, one deduce that

$$\frac{\partial A_y}{\partial z} < 0$$

Computation of $q_{b,1}^*$. From [3.14](#) and $q_{b,0}^*$, one has

$$q_{b,1}^* = \frac{N-2}{N-1} \times \frac{A_y(\sigma_q^2)}{1 + A_y(\sigma_q^2)} \times \frac{S}{N} (I_{s,0} - I_{b,0})$$

Computation of y_k^* The second coefficient κ_2^y of $\Sigma_y^{-1} K_y$ is such that:

$$\begin{aligned} |\Sigma_y| \kappa_2^y &= \left(\sigma_1^2 + \frac{N-1}{N} (1-z) \sigma_2^2 \right) \left(\sigma_1^2 + \frac{N-1}{N} (1 - (1+\alpha)z) \sigma_2^2 \right) \\ &\quad - \left(\sigma_1^2 + \left(1 - \frac{N-2}{N} z\right) \sigma_2^2 \right) \left(\sigma_1^2 + \left(\frac{1}{\alpha} + 1\right) (1-z) \sigma_2^2 \right) \\ &= -\sigma_2^2 \left\{ \frac{1}{N} \left[2 + \frac{(N-1)^2}{N-2} - z \right] \sigma_1^2 + \left[1 + \frac{2}{N\alpha} \right] (1-z) \sigma_2^2 \right\} \end{aligned}$$

Therefore

$$\begin{aligned} \kappa_2^y &= -\frac{\frac{1}{N} \left[2 + \frac{(N-1)^2}{N-2} - z \right] \sigma_1^2 + \left[1 + \frac{2}{N\alpha} \right] (1-z) \sigma_2^2}{\frac{2}{N} \left\{ \left[1 + \frac{1-z}{2(N-2)} \right] \sigma_1^2 + \frac{1-z}{\alpha} \sigma_2^2 \right\}} \\ &= -\frac{\left[1 + \frac{(N-1)^2}{2(N-2)} - \frac{z}{2} \right] \sigma_1^2 + \left[\frac{N}{2} + \frac{1}{\alpha} \right] (1-z) \sigma_2^2}{\left[1 + \frac{1-z}{2(N-2)} \right] \sigma_1^2 + \frac{1-z}{\alpha} \sigma_2^2} \end{aligned}$$

and

$$\begin{aligned}
1 + \kappa_2^y &= \frac{\left[1 + \frac{1-z}{2(N-2)} - 1 - \frac{(N-1)^2}{2(N-2)} + \frac{z}{2}\right] \sigma_1^2 + \left[\frac{1}{\alpha} - \frac{N}{2} - \frac{1}{\alpha}\right] (1-z) \sigma_2^2}{\left[1 + \frac{1-z}{2(N-2)}\right] \sigma_1^2 + \frac{1-z}{\alpha} \sigma_2^2} \\
&= -\frac{N \left[1 - \frac{1}{N} \frac{N-3}{N-2} z\right] \sigma_1^2 + (1-z) \sigma_2^2}{2 \left[1 + \frac{1-z}{2(N-2)}\right] \sigma_1^2 + \frac{1-z}{\alpha} \sigma_2^2}
\end{aligned}$$

Thus, as $0 < z \leq 1$, one has $1 + \kappa_2^y < 0$. Finally set $h_y = -(1 + \kappa_2^y)/N$ to get the result of the proposition.

Variations and bounds of h_y . One can show, denoting $x = \sigma_1^2/\sigma_2^2$, that:

$$\begin{aligned}
\frac{\partial h_y}{\partial z} &= \frac{1 - \left(\frac{1}{N} \frac{N-3}{N-2} x + 1\right) \left(\left[1 + \frac{1-z}{2(N-2)}\right] x + \frac{1-z}{\alpha}\right) + \left(\frac{x}{2(N-2)} + \frac{1}{\alpha}\right) \left(\left[1 - \frac{1}{N} \frac{N-3}{N-2} z\right] x + 1 - z\right)}{2 \left(\left[1 + \frac{1-z}{2(N-2)}\right] x + \frac{1-z}{\alpha}\right)^2} \\
&= \frac{1}{2(N-2)} \frac{x \left(\frac{1}{N} \left(1 - \frac{N-3}{\alpha}\right) + x \left(\frac{1}{2} - \frac{N-3}{N} \frac{N-3/2}{N-2}\right)\right)}{\left(\left[1 + \frac{1-z}{2(N-2)}\right] x + \frac{1-z}{\alpha}\right)^2}
\end{aligned}$$

For $N = 3$, it is clear that $\partial h_y/\partial z > 0$. For $N \geq 4$, one can check that $1 - (N-3)/\alpha < 0$; while $1/2 - \frac{(N-3)(N-3/2)}{N(N-2)} < 0$ for $N \geq 6$. Thus for $N \geq 6$, without ambiguity $\partial h_y/\partial z < 0$. For $N = 4, 5$, there are x_4, x_5 such that for $x < x_N$, $\partial h_y/\partial z < 0$ and for $x > x_N$, $\partial h_y/\partial z > 0$.

Case $N \geq 4$. In this case the partial derivative of h_y with respect to z above is unambiguously negative, thus:

$$h_y \leq \frac{1}{2} \frac{\sigma_1^2 + \sigma_2^2}{\left[1 + \frac{1}{2(N-2)}\right] \sigma_1^2 + \frac{1}{\alpha} \sigma_2^2} = \frac{1}{2} \frac{x + 1}{\frac{N-3/2}{N-2} x + \frac{1}{\alpha}}$$

Computing the derivative of the RHS term with respect to x , it is easy to show that the RHS decreases in x , so that it is maximized for x , and thus

$$0 < h_y \leq \frac{\alpha}{2} \quad \forall N \geq 4$$

Case $N = 3$. The sign of $\partial h_y/\partial z$ is minus the sign of its numerator, which equals $x - 1$. Thus for $N = 3$, $\partial h_y/\partial z < 0$ iff $x > 1$.

And for $N = 3$ inspecting the expression of h_y , it is easy to see that the numerator is smaller than the denominator, so that at least $h_y \leq \frac{1}{2}$.

Variations with respect to x .

$$\begin{aligned} \frac{\partial h_y}{\partial x} &= \frac{1}{2} \frac{\left[1 - \frac{1}{N} \frac{N-3}{N-2} z\right] \left[\left(1 + \frac{1-z}{2(N-2)}\right) x + \frac{1-z}{\alpha}\right] - \left[1 + \frac{1-z}{2(N-2)}\right] \left[\left(1 - \frac{1}{N} \frac{N-3}{N-2} z\right) x + 1 - z\right]}{\left(\left[1 + \frac{1-z}{2(N-2)}\right] x + \frac{1-z}{\alpha}\right)^2} \\ &= \frac{1}{2(N-2)} \times \frac{(1-z) \left(\frac{1}{N} - \frac{1}{2} + \left(\frac{1}{2} - \frac{N-3}{N} \frac{1}{\alpha}\right) z\right)}{\left(\left[1 + \frac{1-z}{2(N-2)}\right] x + \frac{1-z}{\alpha}\right)^2} \end{aligned}$$

The last line has used $1/\alpha - 1 = (1 - \alpha)/\alpha = \frac{(N-1)^{-2}}{1-(N-1)^{-2}} = \frac{1}{N(N-2)}$. As $N \geq 3$, $1/N - 1/2$ is negative, while one can check that $1/2 - \frac{N-3}{N\alpha}$ is positive iff $N \leq 5$. This implies that

- For $N \leq 5$, for z higher than some z_N , h_y increases with x , otherwise it decreases with x .
- For $N \geq 6$, h_y decreases with x .

C.4.5 Proof of proposition 14 (non-manipulable futures)

Here I show that for all $\sigma_1^2 \geq 0$ and all $\sigma_q^2 \geq 0$,

$$A_y(\sigma_1^2, \sigma_q^2) > A(\sigma_1^2, \sigma_q^2)$$

I examine the difference $A - A_y$ and proceed in three steps. First, I show that $A - A_y$ increases with z for any σ_1^2 . Second, I show that for $z = 1$, $A - A_y$ decreases with σ_1^2 . Thus I find that the maximum of $A - A_y$ is attained for $\sigma_1^2 = 0$ and $z = 1 \Leftrightarrow \sigma_q^2 \rightarrow \infty$: I simply show that $A(0, \infty) = A_y(0, \infty)$, and the proposition is proven.

First step. From lemma [7](#), we know that $\frac{\partial A}{\partial z} > 0$, while I compute:

$$\begin{aligned} \frac{\partial A_y}{\partial z} &= \frac{1}{N-1} \left\{ - \left(\frac{\sigma_1^2}{N-2} + \frac{2\sigma_2^2}{\alpha} \right) \left(\left(1 - \frac{z}{(N-1)^2} \right) \sigma_1^2 + (1-z)\sigma_2^2 \right) \right. \\ &\quad \left. + \left(\frac{\sigma_1^2}{(N-1)^2} + \sigma_2^2 \right) \left(\left(2 + \frac{1-z}{N-2} \right) \sigma_1^2 + \frac{2(1-z)}{\alpha} \sigma_2^2 \right) \right\} \\ &\quad \times \left(\left(1 - \frac{z}{(N-1)^2} \right) \sigma_1^2 + (1-z)\sigma_2^2 \right)^{-2} \\ &= - \frac{\sigma_1^4}{N-1} \frac{N^2+4}{(N-1)^2(N-2)} \times \left(\left(1 - \frac{z}{(N-1)^2} \right) \sigma_1^2 + (1-z)\sigma_2^2 \right)^{-2} \\ &< 0 \end{aligned}$$

The antepenultimate line has used $\alpha = \frac{N(N-2)}{(N-1)^2}$. Therefore $\frac{\partial(A-A_y)}{\partial z} > 0$, which concludes the first step.

Second step. Now for $z = 1$,

$$A - A_y = \frac{1}{N-2} \left(\frac{\sigma_1^2 + \sigma_2^2/N}{\sigma_1^2 + \frac{\sigma_2^2}{N-1}} - \frac{2(N-2)}{N-1 - \frac{1}{N-1}} \right)$$

Therefore

$$\frac{\partial(A - A_y)}{\partial \sigma_1^2 / \sigma_2^2} = \frac{\partial A}{\partial \sigma_1^2 / \sigma_2^2} - \frac{1}{N(N-1)} \frac{1}{\left(\sigma_1^2 + \left(1 - \frac{N-2}{N-1} z \right) \sigma_2^2 \right)^2} < 0$$

which proves the second step.

The third step is straightforward.

C.4.6 Proof of corollary [1](#) (non-manipulable futures)

Inspecting the expression h_y , it is clear that as $N \rightarrow \infty$,

$$h_y \rightarrow \frac{1}{2} \frac{\sigma_1^2 + (1-z)\sigma_2^2}{\sigma_1^2 + (1-z)\sigma_2^2},$$

which is bounded. However, it is also clear that A_y converges to zero. Therefore, the equilibrium non-manipulable futures position $y_k^* = \frac{N-2}{N-2} h_f \frac{A_y}{1+A_y} q_k^c$ converges to zero as N diverges.

C.4.7 Proof of theorem 2: $\widehat{W}_{k,0}^y > \widehat{W}_{k,0}^n$

I reexpress the welfare difference as:

$$\begin{aligned}\widehat{W}_{k,0}^y - \widehat{W}_{k,0}^n &= -\gamma\sigma_2^2(q_k^c)^2 \left\{ \left(\frac{A_y}{1+A_y} \right)^2 \left[\frac{x+1-\alpha z}{2} - \tilde{h}_y \left(1 - \frac{\tilde{h}_y}{2} \right) x + \frac{1-z}{\alpha} \tilde{h}_y \left(\alpha - \frac{\tilde{h}_y}{2} \right) \right. \right. \\ &\quad \left. \left. - \left(\frac{A}{1+A} \right)^2 \frac{x+1-\alpha z}{2} \right] \right\} \\ &= -\gamma\sigma_2^2(q_k^c)^2 \left(\frac{A_y}{1+A_y} \right)^2 \frac{x+1-\alpha z}{2} \\ &\quad \times \underbrace{\left(1 - \left(\frac{A}{1+A} \frac{1+A_y}{A_y} \right)^2 - 2 \frac{x+1-z}{x+1-\alpha z} \tilde{h}_y + \frac{x+\frac{1-z}{\alpha}}{x+1-\alpha z} \tilde{h}_y^2 \right)}_{\Phi(\tilde{h}_y)}.\end{aligned}$$

Hence $\widehat{W}_{k,0}^y > \widehat{W}_{k,0}^n$ iff $\Phi(\tilde{h}_y) < 0$. I consider Φ as a second degree polynomial in \tilde{h}_y , taking A and A_y as given. Given that the coefficient in \tilde{h}_y^2 is positive, it is negative iff it has roots $h_- < h_+$ and if the equilibrium value of \tilde{h}_y is in the interval $[h_-, h_+]$. I know check this.

Φ has roots if and only if its discriminant

$$\Delta_{ny} = 4 \frac{x + \frac{1-z}{\alpha}}{x+1-\alpha z} \left[\frac{(x+1-z)^2}{(x+1-\alpha z)(x + \frac{1-z}{\alpha})} - 1 + \left(\frac{A}{1+A} \frac{1+A_y}{A_y} \right)^2 \right]$$

is positive. Now I show that the term in brackets in Δ_{ny} , which determines its sign, decreases with x . It is easy to check that

$$\frac{\partial}{\partial x} \frac{(x+1-z)^2}{(x+1-\alpha z)(x + \frac{1-z}{\alpha})} = \frac{(x+1-z) \left(x + \frac{1-z}{N(N-2)} \right)}{(x+1-\alpha z)^2 (x + \frac{1-z}{\alpha})^2} (\alpha - 1) \leq 0$$

because $\alpha < 1$, and with equality iff $x = 0$ and $z = 0$. In addition, lemma 7 has shown that A decreases with x , thus $A/(1+A)$ also decreases; while A_y increases with x , so $(1+A_y)/A_y$ also decreases with x . Thus the term in brackets decreases with x . It is straightforward to check that as x tends to infinity, the first two terms in brackets cancel out, so that

$$\Delta_{ny} \geq 4 \frac{x + \frac{1-z}{\alpha}}{x+1-\alpha z} \lim_{x \rightarrow \infty} \left(\frac{A}{1+A} \frac{1+A_y}{A_y} \right)^2$$

and given that $\lim_{x \rightarrow \infty} A = \frac{1}{N-2} > 0$ and

$$\lim_{x \rightarrow \infty} A_y = \frac{2 + \frac{1-z}{N-2}}{1 - \frac{z}{N-1}} > 0$$

thus the limit of $\left(\frac{A}{1+A} \frac{1+A_y}{A_y}\right)^2$ is strictly positive. Thus $\Delta_{ny} > 0$, implying that Φ has two real roots. The roots of Φ are

$$h_{\pm} = \frac{x+1-z}{x + \frac{1-z}{\alpha}} \left\{ 1 \pm \sqrt{1 - \left(\frac{x+1-\alpha z}{x+1-z}\right)^2 \left(1 - \left(\frac{A}{1+A} \frac{1+A_y}{A_y}\right)^2\right)} \right\}$$

Given that $h_f \leq 1/2$ (cf. proof of proposition [5](#)), and $h_+ \geq 1$ (in particular $(x+1-z)/(x+(1-z)/\alpha) > 1$), one has $\tilde{h}_y < h_+$.

Now it remains to check that $\tilde{h}_y > h_-$. This is equivalent to showing that

$$\sqrt{1 + \left(\frac{x+1-\alpha z}{x+1-z}\right)^2 \left(\left(\frac{1+A_y^{-1}}{1+A^{-1}}\right)^2 - 1\right)} + \frac{1}{2} \frac{N-2}{N-1} \frac{x + \frac{1-z}{\alpha}}{x+1-z} \frac{\left(1 - \frac{1}{N} \frac{N-3}{N-2} z\right) x + 1 - z}{\left(1 + \frac{1-z}{N-2}\right) x + \frac{1-z}{\alpha}} > 1$$

and given that $\frac{x+1-\alpha z}{x+1-z} > 1$, replacing the ratio by 1 in the above inequality and rearranging, it holds if

$$\frac{1 + A_y^{-1}}{1 + A^{-1}} + \frac{1}{2} \frac{N-2}{N-1} \frac{x + \frac{1-z}{\alpha}}{x+1-z} \frac{\left(1 - \frac{1}{N} \frac{N-3}{N-2} z\right) x + 1 - z}{\left(1 + \frac{1-z}{N-2}\right) x + \frac{1-z}{\alpha}} > 1$$

The left-hand side (LHS) is decreasing in x : the ratio $\frac{1+A_y^{-1}}{1+A^{-1}}$ decreases with x as shown above, and it is easy to check that each non-constant factor in the second term of the decreases with x . Therefore the previous inequality holds for all $x >$

$0, z \in [0, 1)$ if it holds for the limit of the LHS as x becomes infinite, i.e.:

$$\begin{aligned}
& \frac{1 + (N-1) \frac{1 - \frac{z}{(N-1)^2}}{2 + \frac{1-z}{N-2}}}{N-1} + \frac{1}{2} \frac{N-2}{N-1} \frac{1 - \frac{1}{N} \frac{N-3}{N-2} z}{1 + \frac{1-z}{2(N-2)}} > 1 \\
& \frac{2}{N-1} + \frac{1 - \frac{z}{(N-1)^2}}{1 + \frac{1-z}{2(N-2)}} + \frac{N-2}{N-1} \frac{1 - \frac{1}{N} \frac{N-3}{N-2} z}{1 + \frac{1-z}{2(N-2)}} > 2 \\
& \frac{\frac{N-1}{N-2} + 1 - \left(\frac{1}{(N-2)(N-1)} + \frac{N-3}{N(N-2)} \right) z}{1 + \frac{1-z}{2(N-2)}} > 2 \\
& \frac{N-1}{N-2} + 1 - \frac{1}{N-2} \left(\frac{1}{N-1} + \frac{N-3}{N} \right) z > 2 + \frac{1}{N-2} - \frac{z}{N-2} \\
& \left(\frac{1}{N-1} + \frac{N-3}{N} \right) z < z
\end{aligned}$$

The latter inequality always hold for $z > 0$, since $\frac{1}{N-1} + \frac{N-3}{N} = 1 - \frac{2N-3}{N(N-1)} < 1$.

From this I conclude that $\tilde{h}_y > h_-$, and the theorem is proven.