Control and Out-of-Sample Validation of Dependent Risks

Christian, Gourieroux∗and Wei, Liu†

May 25, 2007

Abstract

The aim of this paper is to propose a methodology for validating the reserve levels proposed by a bank for one or several risky budget lines. We emphasize the importance of the validation criterion and of its conditional interpretation. We also introduce and compare different methodologies for reallocating the reserves among several budget lines.

Keywords: Risk Control, Predictive Ability, VaR, Validation, Inequality Hypothesis, Misspecification.

JEL Classification: C22, C52, C53

∗CREST and University of Toronto.
†University of Toronto and Fitch Ratings.
1 Introduction

The current regulation introduced in Finance by Basel I and Basel II (or in Insurance by Solvency II) has at least three limitations: i) The first one is the choice of the so-called Value-at-Risk (VaR) to measure the risk and to fix the required capital. Indeed, the VaR accounts for the probability of a loss, but not for the expected magnitude of this loss (as the TailVaR), or for the possibility of extreme losses (as distortion risk measures). ii) The second limitation is the lack of coherency between the suggested computation of the VaR as a conditional quantile and the ex-post validation of this computation, which uses unconditional (historical) hedging errors. iii) Finally, the computation of the VaRs are performed independently on different lines of financial products, without taking into account the possible dependence of the risks between lines.

The aim of this paper is to propose solutions to the two last limitations by focusing on the (implicit) decision criterion of the internal or external validator. For expository purpose, we follow the practice of measuring the risk by means of the VaR. In Section 2, we focus on a single line of financial products, that is on a single risk. We introduce the decision criterion which underlies the choice of a VaR and discuss the criterion value when a suboptimal control variate is selected. The criterion value can be decomposed into its minimal attainable value plus a term measuring the lack of optimality. In a perspective of measuring the out-of-sample performances of a proposed control variate, we analyze the joint dynamic of these two components. The case of several lines of financial products is considered in Section 3. We introduce a decision criterion function, which extends naturally the criterion considered for a single budget line. By optimizing this criterion, with or without constraint on the global risk, we get the allocations of reserves by budget lines. Section 4 concludes. Proofs are gathered in appendices.

2 Conditional risk control

It is emphasized in the recent literature [see e.g. Hansen (2004), Giacomini and White (2006)] that the “out-of-sample comparison of predictive ability has to be based on inference about conditional expectations of forecast and forecast errors rather than their unconditional expectations”. The same remark applies for the validation by the regulators of the approaches followed by banks and insurance companies to fix their reserves. This section discusses this principle for the computation of the VaR and the analysis of its out-of-sample performances.

2.1 The underlying criterion

The approach is based on the underlying criterion followed by regulators to derive the VaR, that is a conditional quantile of the future portfolio value. Let us denote by $y_t$ the opposite of the portfolio value at date $t$, by $I_t$ the information available at time $t$, which includes at least the current and lagged values of $y$ and by $z_t$ the level of reserve, which can be considered as a control variate. This level is proposed by the bank according to the information $I_t$. The standard decision criterion is:

$$
\Psi(t, z_t) = E_t \left[ \alpha (y_{t+1} - z_t)^+ + (1 - \alpha) (y_{t+1} - z_t)^- \right],
$$

(2.1)
where $E_t$ denotes the conditional expectation given $I_t$, $y^+ = \max(y, 0)$, $y^- = \max(-y, 0)$, and $\alpha$ is
the announced risk level. This decision criterion corresponds to an asymmetric loss function (written
in §). Two types of errors can arise. The amount of reserve $z_t$ can be too small, implying a loss $y_{t+1} - z_t$; it can also be too large, which implies an amount of money $z_t - y_{t+1}$ not used for profitable
investment. The criterion is weighting both types of errors, with more weights to avoid the first type
of error from the prudential (regulator) point of view.\footnote{Since the quantile concerns the loss and profit variable, $\alpha$ is large in practice. Typically, it is equal to 95%, i.e. to a
level of 5% when the problem is written in terms of profit and loss.}

Let us denote $f_t$ and $F_t$ the conditional pdf and cdf of $y_{t+1}$, respectively. It is well-known [see e.g.
Koenker (2005)], that the optimal value of the control variate:

$$
\hat{z}_t = \arg \min_{z_t \in I_t} E_t \left[ \alpha (y_{t+1} - z_t)^+ + (1 - \alpha) (y_{t+1} - z_t)^- \right],
$$

is the conditional quantile at risk level $\alpha$:

$$
\hat{z}_t = F_t^{-1}(\alpha).
$$

However, there exists a large number of decision criteria providing the conditional quantile as the
optimal control variate. It has been shown in [Gourieroux and Monfort (1995), Vol 1, Section 8.5.9],
that these criteria can be written as:

$$
\Psi(t, z_t) = E_t \left[ \alpha \left[ A(y_{t+1}) - A(z_t) \right]^+ + (1 - \alpha) \left[ A(y_{t+1}) - A(z_t) \right]^- \right],
$$

where $A$ is an increasing function. In practice, it is important to choose among them, a criterion with
clear financial interpretation, that is weighting appropriately the loss magnitude.\footnote{In this respect, the decision criterion introduced in [Hansen (2004), Example 2, neglects the magnitude of the loss.}

For any control variate, the criterion value can be decomposed as:

$$
\Psi(t, z_t) = \Psi(t, \hat{z}_t) + \left[ \Psi(t, z_t) - \Psi(t, \hat{z}_t) \right],
$$

where the second component of the right hand side measures the lack of optimality. For the standard
objective function, where $A$ is the identity function, this measure is equal to [see Appendix A]:

\[
c(t, z_t) = \int_{z_t}^{\hat{z}_t} \left[ F_t(\hat{z}_t) - F_t(y) \right] dy
\]

(2.6)

\[
= \begin{cases} 
E_t \left[ (y_{t+1} - z_t)^\mathbb{1}_{(z_t < y_{t+1} < \hat{z}_t)} \right], & \text{if } \hat{z}_t > z_t, \\
E_t \left[ (z_t - y_{t+1})^\mathbb{1}_{(\hat{z}_t < y_{t+1} < z_t)} \right], & \text{if } \hat{z}_t \leq z_t.
\end{cases}
\]

(2.7)

In particular, when the selected control variate is close to the optimal one $z_t \simeq \hat{z}_t$, we get the expansion:

\[
c(t, z_t) \simeq \frac{1}{2} f_t(\hat{z}_t)(\hat{z}_t - z_t)^2 = \frac{1}{2} f_t \left[ F_t^{-1}(\alpha) \right] (\hat{z}_t - z_t)^2.
\]

(2.8)

The first component of decomposition (2.5) is equal to:

\[
\Psi(t, \hat{z}_t) = \alpha P_t \left[ y_{t+1} > \hat{z}_t \right] E_t \left[ (y_{t+1} - \hat{z}_t)^+ \left| y_{t+1} > \hat{z}_t \right. \right]
+ (1 - \alpha) P_t \left[ y_{t+1} < \hat{z}_t \right] E_t \left[ (y_{t+1} - \hat{z}_t)^- \left| y_{t+1} < \hat{z}_t \right. \right]
= \alpha (1 - \alpha) \left[ \text{TailVaR}^+ (\alpha) + \text{TailVaR}^- (\alpha) \right],
\]

(2.9)

where the TailVaR measures the "expected loss" when a loss occurs. This expression highlights the importance of the TailVaR in the validating criterion even if it is not the optimal level of the control variate. Thus, if the optimal control is applied, the regulator needs to follow up both the VaR for the control variate, and the TailVaR, for the objective function.

### 2.2 Dynamic of the decision criterion

It is now possible to study the consequence on the criterion of any proposed methodology to compute the control variate. More precisely, for any methodology $z_t$, we can consider the joint dynamics of $\Psi(t, \hat{z}_t)$ and $c(t, z_t)$.

As an illustration, let us assume that the dynamics of $y_{t+1}$ is such that:

\[
y_{t+1} = m_t + \sigma_t u_{t+1},
\]

(2.10)

where $m_t, \sigma_t$ are functions of the past and $(u_t)$ is a sequence of i.i.d. error terms. This specification includes the ARMA-GARCH model, where the information set corresponds to the current and lagged values of $y$, but also the stochastic mean and volatility model, where the information set also includes the current and lagged values of the factors driving the stochastic parameters. Let us denote by $Q$ the quantile function of the standardized error $u$, $H(x) = E(u - x)^+$, and $E(u) = 0$. The conditional quantile is:

\[
\hat{z}_t = m_t + \sigma_t Q(\alpha).
\]

(2.11)
Thus, we deduce:

\[
\Psi(t, \hat{z}_t) = \alpha E_t \left[ (y_{t+1} - \hat{z}_t)^+ \right] + (1 - \alpha) E_t \left[ (y_{t+1} - \hat{z}_t)^- \right]
\]

\[
= \alpha \sigma_t E_t \left[ (u_{t+1} - Q(\alpha))^+ \right] + (1 - \alpha) \sigma_t E_t \left[ (u_{t+1} - Q(\alpha))^- \right]
\]

\[
= \sigma_t \delta(\alpha),
\]

(2.12)

where

\[
\delta(\alpha) = H \left[ Q(\alpha) \right] + (1 - \alpha) Q(\alpha).
\]

(2.13)

Let us now assume that the VaR proposed by the bank is computed by historical simulation (on a large window), that is, by considering the unconditional quantile. Then, we get \( z_t = z \), independent of the past, and the value of the criterion becomes:

\[
\Psi(t, z) = \alpha E_t \left[ (y_{t+1} - z)^+ \right] + (1 - \alpha) E_t \left[ (y_{t+1} - z)^- \right]
\]

\[
= E_t \left[ (y_{t+1} - z)^+ \right] - (1 - \alpha) \left( E_t [y_{t+1}] - z \right)
\]

\[
= \sigma_t H \left[ \frac{z - m_t}{\sigma_t} \right] - (1 - \alpha) (m_t - z).
\]

(2.14)

The correcting component is:

\[
c(t, z_t) = \sigma_t \left[ H \left( \frac{z - m_t}{\sigma_t} \right) - H \left( Q(\alpha) \right) \right] - (1 - \alpha) \left[ m_t + \sigma_t Q(\alpha) - z \right],
\]

(2.15)

and its local expansion becomes (see Appendix B):

\[
c(t, z_t) = \frac{1}{2\sigma_t} g \left[ Q(\alpha) \right] (m_t + \sigma_t Q(\alpha) - z)^2,
\]

(2.16)

where \( g \) is the pdf of the error term \( u \).

This example shows that:

1. The minimal attainable criterion value is a stochastic function of the scale factor \( \sigma_t \) only;

2. The measure of the lack of optimality of the unconditional VaR depends on both the location and scale factors \( m_t, \sigma_t \); therefore, the two components of the decision criterion have dependent joint dynamics;

3. The same remark applies for the relative lack of optimality \( c(t, z_t) / \Psi(t, \hat{z}_t) \);

4. The optimal criterion value is highly volatile, since it is proportional to \( \sigma_t \). This is a consequence of the usual choice of a fixed risk level \( \alpha \) by validators instead of a path dependent risk level \( \alpha_t \). This feature has already been noted when discussing how to take into account the cycle effect in the regulation, that is the point-in-time (PIT) versus the through-the-cycle (TTC) methodologies. In this framework, it has been suggested to adjust the risk level along the cycle. Similarly, the risk level could be adjusted to the volatility level.
As an illustration, let us consider the stochastic volatility model:

\[ r_t = -y_t = 0.5\sigma_t^2 + \sigma_t \varepsilon_t, \tag{2.17} \]

where the volatility \( \sigma_t^2 \) follows an autoregressive gamma (ARG) process with parameters \( \rho = 0.96, \delta = 3.6, c = 7.34 \times 10^{-3} \) \cite{GourierouxJasiak2006a}. The historical parameters \( \rho, \delta, c \) are such that the ARG volatility process matches the stationary mean, variance and first-order autocorrelation of the discretely sampled Cox-Ingersoll-Ross process estimated by \cite{Andersenetal2002}. The dynamic model is an extension of the standard \cite{BallRoma1994} model, including a risk premium. We provide in Figure 1 a joint simulated path of \((r_t, \sigma_t^2), t = 1, \ldots, 250\), corresponding to solid line and dashed line, respectively.

This path is used to estimate the historical (unconditional) distribution of the return and to deduce its (estimated) unconditional quantile at \( 1 - \alpha = 5\% \). This unconditional quantile is equal to \( \tilde{z} = 2.31 \). Figure 2 displays the evolution of the conditional quantile \( \hat{z}_t \) (solid line) and compares with the unconditional quantile (dashed line) \( \hat{z} \). The simulated path of \( r_t \) (shorter dashed line) is also plotted to reveal the relative locations of the estimated quantiles. As expected the unconditional quantile is above the average of the conditional quantiles. This represents the cost of not taking into account the largest information when evaluating the risk.

For a suboptimal control, both components of the objective function have to be analyzed jointly. We provide in Table 1 the historical summary statistics of the series, that are their mean, variance, skewness, kurtosis.

\[ \text{Table 1: Historical Summary Statistics}. \]

The distribution of the loss function \( \Psi(t, z) \), when the suboptimal unconditional quantile is used, features strong positive skewness and very fat tails. The fat right tail arises since the unconditional VaR estimate tends to underestimate the TailVaR when the volatility is large. This underestimation of TailVaR is further amplified by the volatility level as specified by equation (2.14). This effect is

\[^3\text{The conditional quantile is computed with the value of the parameters used in the simulation study. In practice these values are unknown and have to be estimated, which induces an additional uncertainty.}\]
significantly smaller when the conditional quantile is used, and the kurtosis is four times smaller.

Cross-autocorrelations of the series and their squares are provided in Table 2. \( \rho(k) \) denotes the kth order autocorrelation matrix. For instance,

\[
\rho(3) = \text{corr} \left( \{ \psi(t, \hat{z}_t), c(t, z_t), \psi(t, z) \}' \right) .
\]

By construction, the serial correlations on the optimal criterion value \( \Psi(t, \hat{z}_t) \) and their squares are equal to the serial correlations of \( \sigma_t \) and \( \sigma_t^2 \). Thus, it is not surprising to observe the long memory effect corresponding to the high persistence of volatility. A smaller persistence can be observed on the criterion value and its squares, when unconditional quantile is used as control variate.

### 2.3 Comparing the methodologies proposed to compute the reserves

The aim of this section is to compare different methodologies \( z_{1,t}, z_{2,t} \), say, used to compute the reserve corresponding to a same risky portfolio and a same announced risk level. First of all, two methodologies cannot be ranked unambiguously in general; indeed, methodology 1 can be preferred in some environments (such as expansion periods, or high volatility periods), whereas methodology 2 can be preferable in other environments (such as recession periods, or low volatility periods).

However, it is possible to develop tractable diagnostic concerning properties valid for all environments, for instance to test the hypothesis:

\[
H_0 : \text{ ( methodology 1 is better than methodology 2 for any environment. )}
\]

The null hypothesis is:

\[
H_0 : \left[ \Psi(t, z_{1,t}) \leq \Psi(t, z_{2,t}) \right], \text{ for any value of the conditioning variable},
\]

and involves an infinite number of conditional inequality moment restrictions. This hypothesis can be written equivalently in terms of unconditional inequality moment restrictions by introducing instruments \( g_t \) as:

\[
H_0 : \left[ E \left\{ g_t \left[ \alpha(y_{t+1} - z_{1,t})^+ + (1 - \alpha)(y_{t+1} - z_{1,t})^- \right] \right\} \leq E \left\{ g_t \left[ \alpha(y_{t+1} - z_{2,t})^+ + (1 - \alpha)(y_{t+1} - z_{2,t})^- \right] \right\}, \right.
\]

for any positive function \( g_t \) depending on information \( I_t \).

At this step, it is important to mention that the standard diagnostics, proposed in the literature [see e.g. Diebold and Mariano (1995), Harvey et al. (1997a), or the survey by Stock and Watson (2005)] or used by practitioners, validators, or regulators, are based on the unconditional loss function only;
that is, select $g_t = 1$ as the single instrumental variable.\(^4\) The recent literature has emphasized the importance of selecting a larger number of appropriate instruments. For instance, Gourieroux and Jasiak (2006) considered lagged values of the loss function, such as:

$$g_{j,k,t} = \alpha((y_{t+1} - z_{j,t} - k)^+ + (1 - \alpha)(y_{t+1} - z_{j,t} - k)^-), j = 1, 2.$$  

By considering the single unitary instrument, we focus on the historical average loss. By choosing lagged values of the objective function as instruments, we consider also the possibility of loss clustering, that is of a large cumulated loss on consecutive periods of time. When a finite number of instruments $g_{1,t}, \ldots, g_{K,t}$, say, are selected, the hypothesis concerns $K$ positivity restrictions on unconditional moments. It can be tested by Wald type procedures using the results established in Kudô (1963), Gourieroux et al. (1980, 1982), Wolak (1987), and Wolak (1991).

The procedure is as follows:

1. Approximate each unconditional moment by its sample counterpart. Namely, the following moment:

$$\theta_k = E \left[ g_{k,t} \left\{ \alpha \left( (y_{t+1} - z_{2,t})^+ - (y_{t+1} - z_{1,t})^+ \right) + (1 - \alpha) \left( (y_{t+1} - z_{2,t})^- - (y_{t+1} - z_{1,t})^- \right) \right\} \right], k = 1, \ldots, K,$$

is estimated by:

$$\tilde{\theta}_{k,T} = \frac{1}{T} \sum_{t=1}^{T-1} g_{k,t} \left\{ \alpha \left( (y_{t+1} - z_{2,t})^+ - (y_{t+1} - z_{1,t})^+ \right) + (1 - \alpha) \left( (y_{t+1} - z_{2,t})^- - (y_{t+1} - z_{1,t})^- \right) \right\}, k = 1, \ldots, K.$$

2. Estimate the asymptotic variance-covariance matrix $\hat{\Omega}_T$ of $\hat{\theta}_T = (\hat{\theta}_{1,T}, \ldots, \hat{\theta}_{K,T})'$, by the sample variance of the vector of sample averages, given by,

$$\hat{\Omega}_T = \hat{\Gamma}_T(0) + \sum_{\tau=-L}^{L} \hat{\Gamma}_T(\tau),$$

where $\hat{\Gamma}_T(\tau)$ is the sample counterpart of the autocovariance matrix at lag $\tau$ of the multivariate process $g_t \left\{ \alpha \left( (y_{t+1} - z_{2,t})^+ - (y_{t+1} - z_{1,t})^+ \right) + (1 - \alpha) \left( (y_{t+1} - z_{2,t})^- - (y_{t+1} - z_{1,t})^- \right) \right\}$ and $L$ defines the bandwidth.

\(^4\)Direct comparisons of the proposed and optimal VaR have the same drawback [see e.g. Mancini and Trojani (2005)].
3. Compute the inequality constrained estimator $\hat{\theta}_T^0$ of $\theta$ as the solution of:

$$\begin{align*}
\hat{\theta}_T^0 &= \arg \min_{\theta : \theta_k \geq 0, k = 1, \ldots, K} \left( \hat{\theta}_T - \theta \right) \Omega_T^{-1} \left( \hat{\theta}_T - \theta \right).
\end{align*}$$

4. Compute the test statistics as the standardized sum of squared residuals:

$$SSR_T = \left( \hat{\theta}_T - \hat{\theta}_T^0 \right) \Omega_T^{-1} \left( \hat{\theta}_T - \hat{\theta}_T^0 \right).$$

5. Reject the null hypothesis $H_0$ if $SSR_T > c$, where the critical value with type-I error 5% is the 95% quantile of a mixture of chi-square distributions $\sum_{k=0}^{K} \pi_k \chi^2(k)$, where the weights depend on the pattern of the $\Omega$ matrix. For instance, if $K = 2$, we have

$$\pi_0 = \frac{1}{2 \pi \cos^{-1}(\rho(\theta_1, \theta_2))} \cdot \frac{1}{2}, \quad \pi_1 = \frac{1}{2}, \quad \text{and} \quad \pi_2 = \frac{1}{2} - \pi_0,$$

where $\rho(\theta_1, \theta_2)$ is the correlation between $\theta_1$ and $\theta_2$.

As an illustration, let us continue the example of Section 2.2 with the suboptimal unconditional quantile $z_{1t} = \hat{z}$, and the optimal conditional quantile $z_{2t} = \hat{z}_t$. The test procedure can be applied with the following instruments:

Case I: $g_{1t} = 1$;

Case II: $g_{1t} = 1, g_{2t} = |r_t - \hat{z}|$;

Case III: $g_{1t} = 1, g_{3t} = |r_t - \hat{z}_{t-1}|$;

Case IV: $g_{1t} = 1, g_{2t} = |r_t - \hat{z}|, g_{3t} = |r_t - \hat{z}_{t-1}|$.

Case I is the standard practice of historical averaging. Case II (resp. Case III) includes also an instrument to capture clustering effect for the suboptimal control (resp. optimal control). Case IV gathers all instruments. The testing procedures are applied for $T = 250$ (i.e. one year of daily data) and $\hat{\Omega}_T$ is computed assuming $L = 5$. The results of the test are given in Table 3, where $Y$ means that the test concludes that the estimated optimal procedure $\hat{z}_t$ is better than the suboptimal procedure $\hat{z}$. When $K \leq 2$, the critical values $c$ are computed by using the closed form expressions provided by Gourieroux et al. (1982). For case IV, the weights $\pi_k$ are obtained through a simulation procedure [see e.g. Wolak (1987)]. As expected, the optimal procedure is proved to be better than the suboptimal procedure in all cases. The comparison of the methodologies based on conditional and unconditional VaR is less clear, if we follow the standard validation practice, which counts the number of exceedances. Indeed, the numbers of times that the observed returns exceed the estimated VaRs are similar in our example for both models.

[Insert Table 3 Test results].

The example above has just been given to illustrate the power of the testing procedure, since in practice the competing control variates $z_{1t}, z_{2t}$ have generally no optimal properties and the validator has only partial knowledge on the way they have been computed.
3 Dependent risks

This section considers the joint analysis of two lines of financial products. The method has to account for the possible dependence between the extreme risks associated with the two lines. First, we consider this question by focusing on an appropriate criterion function, and, as a by-product, derives a notion of bidimensional quantile. Then, we discuss the problem of reserve allocation when the global reserve is fixed. The validation approach introduced in Section 2 can be easily extended to this framework.

3.1 The decision criterion

Let us now consider two lines of financial products with loss and profit values $y_{j,t}$, $j = 1, 2$, and denote $z_{j,t}$, $j = 1, 2$ the associated amount of reserves. The criterion function can be generalized to:

$$
\Psi(t, z_{1,t}, z_{2,t}) = E_t \left[ \alpha_{11}(y_{1,t+1} - z_{1,t})^+ (y_{2,t+1} - z_{2,t})^- + \alpha_{12}(y_{1,t+1} - z_{1,t})^+ (y_{2,t+1} - z_{2,t})^- 
+ \alpha_{21}(y_{1,t+1} - z_{1,t})^- (y_{2,t+1} - z_{2,t})^+ + \alpha_{22}(y_{1,t+1} - z_{1,t})^- (y_{2,t+1} - z_{2,t})^- \right],
$$

(3.1)

where $\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}$ are positive weights. By introducing the cross-products, the loss function accounts for all types of joint losses on the two product lines.

If $\alpha_{11} = \alpha^2, \alpha_{12} = \alpha_{21} = \alpha(1 - \alpha), \alpha_{22} = (1 - \alpha)^2$, the criterion becomes:

$$
\Psi(t, z_{1,t}, z_{2,t}) = E_t \left\{ \alpha(y_{1,t+1} - z_{1,t})^+ (1 - \alpha)(y_{1,t+1} - z_{1,t})^- \right\} \left\{ \alpha(y_{2,t+1} - z_{2,t})^+ (1 - \alpha)(y_{2,t+1} - z_{2,t})^- \right\},
$$

(3.2)

or, equivalently,

$$
\Psi(t, z_{1,t}, z_{2,t}) = E_t \left[ \alpha(1 - \alpha)(y_{1,t+1} - z_{1,t})^- \right] E_t \left[ \alpha(1 - \alpha)(y_{2,t+1} - z_{2,t})^- \right] + \text{Cov} \left\{ \alpha(y_{1,t+1} - z_{1,t})^+ (1 - \alpha)(y_{1,t+1} - z_{1,t})^-, \alpha(y_{2,t+1} - z_{2,t})^+ (1 - \alpha)(y_{2,t+1} - z_{2,t})^- \right\}.
$$

(3.3)

The first component is the product of criterion values for a separate analysis of the two product lines, whereas the second term emphasizes the effect of the possible dependence between risks.

**Remark 1.** As in Section 2, alternative decision criteria can be introduced by weighting differently...
the losses according to their magnitudes. For instance, we can consider the criterion:

\[
\Psi(t, \tilde{z}_1, \tilde{z}_2; A_1, A_2) = E_t \left\{ \alpha_{11} [A_1(y_{1,t+1}) - A_1(z_{1,t})]^+ [A_2(y_{2,t+1}) - A_2(z_{2,t})]^+ + \alpha_{12} [A_1(y_{1,t+1}) - A_1(z_{1,t})]^+ [A_2(y_{2,t+1}) - A_2(z_{2,t})]^- + \alpha_{21} [A_1(y_{1,t+1}) - A_1(z_{1,t})]^-[A_2(y_{2,t+1}) - A_2(z_{2,t})]^+ + \alpha_{22} [A_1(y_{1,t+1}) - A_1(z_{1,t})]^-[A_2(y_{2,t+1}) - A_2(z_{2,t})]^-, \right\},
\]

where \(A_1, A_2\) are two increasing functions.

## 3.2 The optimal control variates

The optimal control variates are solutions of the first-order conditions (note that the objective function is convex); we get:

\[
\begin{align*}
\frac{\partial \Psi}{\partial \tilde{z}_1}(t, \tilde{z}_1, \tilde{z}_2; t) &= 0, \\
\frac{\partial \Psi}{\partial \tilde{z}_2}(t, \tilde{z}_1, \tilde{z}_2; t) &= 0,
\end{align*}
\]

where:

\[
\frac{\partial \Psi}{\partial \tilde{z}_1}(t, \tilde{z}_1, \tilde{z}_2; t) = E_t \left\{ - \mathbb{1}_{y_{1,t+1} > z_{1,t}} \left[ \alpha_{11}(y_{2,t+1} - z_{2,t})^+ + \alpha_{12}(y_{2,t+1} - z_{2,t})^- \right] \right\} + E_t \left\{ \mathbb{1}_{y_{1,t+1} < z_{1,t}} \left[ \alpha_{21}(y_{2,t+1} - z_{2,t})^+ + \alpha_{22}(y_{2,t+1} - z_{2,t})^- \right] \right\},
\]

and a similar expression for \(\frac{\partial \Psi}{\partial \tilde{z}_2}(t, \tilde{z}_1, \tilde{z}_2; t)\).

The system of first-order conditions has no closed form solution in general. However, it can be easily solved recursively when \(\alpha_{11} = \alpha^2, \alpha_{12} = \alpha_{21} = \alpha(1 - \alpha), \alpha_{22} = (1 - \alpha)^2\), since the criterion function is easily concentrated with respect to \(z_{1,t}\) (resp. \(z_{2,t}\)). Indeed, let us assume that \(z_{2,t}\) is given. We have:

\[
\Psi(t, \tilde{z}_1, \tilde{z}_2; t) = E_t \left\{ (y_{1,t+1} - z_{1,t})^+ E \left[ \alpha^2(y_{2,t+1} - z_{2,t})^+ + \alpha(1 - \alpha)(y_{2,t+1} - z_{2,t})^- \right| y_{1,t+1}, I_t \right\} + (y_{1,t+1} - z_{1,t})^- E \left[ \alpha(1 - \alpha)(y_{2,t+1} - z_{2,t})^+ + (1 - \alpha)^2(y_{2,t+1} - z_{2,t})^- \right| y_{1,t+1}, I_t \right\} \]

\[
= E_t \left\{ \delta(y_{1,t+1}, t) \left[ \alpha(y_{1,t+1} - z_{1,t})^+ + (1 - \alpha)(y_{1,t+1} - z_{1,t})^- \right] \right\},
\]

where

\[
\delta(y_{1,t+1}, t) = E_t \left[ \alpha(y_{2,t+1} - z_{2,t})^+ + (1 - \alpha)(y_{2,t+1} - z_{2,t})^- \right| y_{1,t+1}, I_t \right].
\]
Thus, $\Psi(t, z_{1,t}, z_{2,t})$ is proportional to

$$E_t^{Q_t^1} \left[ \alpha(y_{1,t+1} - z_{1,t})^+ + (1 - \alpha)(y_{1,t+1} - z_{1,t})^- \right],$$

where $Q_t^1$ denotes the probability deduced from the historical one by applying a change of density proportional to $\delta(y_{1,t+1}, t)$. This modified probability depends on the methodology $z_{2,t}$. Since a similar approach can be used to concentrate with respect to $z_{2,t}$, the optimal strategies can be derived recursively. At step $p$, the strategies are $z_{1,t}^{(p)}, z_{2,t}^{(p)}$, say. They are used to derive the two modified probabilities $Q_t^{1,p}$ and $Q_t^{2,p}$. Then $z_{1,t}^{(p+1)}$, $z_{2,t}^{(p+1)}$ are the $\alpha$-quantiles of the modified conditional probabilities $Q_t^{1,p}$ and $Q_t^{2,p}$.

Thus, even for this situation when the weights are restricted by $\alpha_{11} = \alpha^2, \alpha_{12} = \alpha_{21} = \alpha(1 - \alpha), \alpha_{22} = (1 - \alpha)^2$, the distributions have to be changed for computing the quantile, and the optimal strategy does not correspond to the standard choice of VaR performed separately for each product line.

### 3.3 Reserve allocation under global constraint

The external regulator controls the risk on some aggregated budget line. For internal control, the management of the bank has to allocate this total reserve among the different sublines. We will consider a global line composed of two sublines $y_{1,t}, y_{2,t}$. The disaggregate reserves are $z_{1,t}, z_{2,t}$, whereas the total reserve is $z_t = z_{1,t} + z_{2,t}$. If we retain the criterion function (3.1), the optimization problem becomes:

$$\begin{align*}
\min_{z_{1,t}, z_{2,t}} & \Psi(t, z_{1,t}, z_{2,t}), \\
\text{s.t.} & \ z_{1,t} + z_{2,t} = z_t.
\end{align*}$$

The restriction on the total reserve can be solved to get the new optimization problem:

$$\min_{z_{1,t}} \ E_t \left[ \alpha_{11} \ (y_{1,t+1} - z_{1,t})^+ + (y_{2,t+1} - z_{1,t})^+ + \alpha_{12} \ (y_{1,t+1} - z_{1,t})^- + (y_{2,t+1} - z_{1,t})^- + \alpha_{21} \ (y_{1,t+1} - z_{1,t})^- + (y_{2,t+1} - z_{1,t})^- - (y_{2,t+1} - z_{1,t})^- \right].$$

(3.7)

The solutions of this problem will be denoted by $\hat{\hat{z}}_{j,t}(z_t, \alpha), j = 1, 2$, where $\alpha = (\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22})$. Thus, $\hat{\hat{z}}_{j,t}(z_t, \alpha)$ solves

$$\begin{align*}
\frac{\partial \Psi}{\partial z_t} (t, \hat{\hat{z}}_{1,t}(z_t, \alpha), z_t - \hat{\hat{z}}_{1,t}(z_t, \alpha)) &= 0, \\
\hat{\hat{z}}_{2,t}(z_t, \alpha) &= z_t - \hat{\hat{z}}_{1,t}(z_t, \alpha),
\end{align*}$$

(3.8)
where:

\[
\frac{\partial \Psi}{\partial z_{1}}(t, z_{1,t}, z_{t} - z_{1,t}) = E_{t} \left\{ \left[ \mathbb{1}_{y_{1,t+1} > z_{1,t}} \mathbb{1}_{y_{2,t+1} > z_{1,t}} - \mathbb{1}_{y_{1,t+1} > z_{1,t}} \mathbb{1}_{y_{2,t+1} < z_{t} - z_{1,t}} \right] \alpha_{11} - \mathbb{1}_{y_{1,t+1} < z_{1,t}} \mathbb{1}_{y_{2,t+1} < z_{t} - z_{1,t}} \alpha_{12} - \mathbb{1}_{y_{1,t+1} < z_{1,t}} \mathbb{1}_{y_{2,t+1} > z_{1,t}} \alpha_{21} + \mathbb{1}_{y_{1,t+1} < z_{1,t}} \mathbb{1}_{y_{2,t+1} < z_{t} - z_{1,t}} \alpha_{22} \right\} (y_{1,t+1} - y_{2,t+1} + z_{t} - 2z_{1,t}) \right\}.
\]

### 3.4 Comparison of the methodologies

By comparing the standard approach and the new methodologies introduced in Sections 3.2, 3.3, we can distinguish three ways of fixing the reserves $z_{1,t}, z_{2,t}$.

**Method 1:** Fix a risk level $\alpha^{*}$ and compute the disaggregate reserve levels by $\hat{z}_{j,t}(\alpha^{*}) = F_{j,t}^{-1}(\alpha^{*})$, where $F_{j,t}$ is the conditional cdf of $y_{j,t}$.

**Method 2:** Apply the optimization problem (3.11) without restrictions on $z_{1,t}, z_{2,t}$. We get solutions $\hat{z}_{j,t}(\alpha), j = 1, 2$, say, which depend on $\alpha = (\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22})$.

**Method 3:** Apply a two-step approach. First fix a risk level $\alpha^{*}$ for the aggregated line and compute $\hat{z}_{t}(\alpha^{*}) = F_{t}^{-1}(\alpha^{*})$, where $F_{t}$ is the conditional cdf of the aggregate line $y_{t} = y_{1,t} + y_{2,t}$. Then, look for solutions of the constrained optimization problem, that are $\hat{z}_{j,t}(\alpha^{*}, \alpha), j = 1, 2$.

The three approaches above depend on different risk level parameters, that are $\alpha^{*}$ for method 1, $\alpha$ for method 2, and $(\alpha^{*}, \alpha)$ for method 3. They are difficult to compare in a general setting. We discuss below the reserve allocation for specific weighting functions and return distributions.

**i) Unconstrained methods**

As mentioned above, methods 1 and 2 coincide if $\alpha_{11} = (\alpha^{*})^{2}, \alpha_{22} = (1 - \alpha^{*})^{2}, \alpha_{12} = \alpha_{21} = \alpha^{*}(1 - \alpha^{*})$, and if the disaggregate risks are independent. In the sequel, we select these weights, but do not assume independent risks. To highlight the differences between methods 1 and 2, let us consider the case of i.i.d. Gaussian variables $(y_{1,t}, y_{2,t}) \sim N(m_{1}, m_{2}, \sigma_{1}^{2}, \rho \sigma_{1} \sigma_{2}, \sigma_{2}^{2})$. It is equivalent to write $y_{j,t} = m_{j} + \sigma_{j} u_{j,t}, j = 1, 2$, where $(u_{1,t}, u_{2,t}) \sim N(0, 0, 1, \rho)$. It is easily checked that the reserves computed with $y_{j,t}$ for both methods 1 and 2 are deduced from the reserves computed with $u_{j,t}, j = 1, 2$ by the same affine transformations $u_{j} \rightarrow m_{j} + \sigma_{j} u_{j}$. Thus, without loss of generality, we can assume $m_{1} = m_{2} = 0, \sigma_{1}^{2} = \sigma_{2}^{2} = 1$; then, the solutions satisfy $z_{1,t}(\alpha) = z_{2,t}(\alpha)$ for both methods. We provide in Figure 3 the pattern of the disaggregate reserve level as function of correlation $\rho$, for $\alpha^{*} = 95\%$. The reserve level corresponding to $\rho = 0$ corresponds to the standard derivation line per line. The horizontal line is 1.645, representing the 95th percentile of the standard normal distribution. In order to reduce the effect of sampling errors, we choose sample size to be 20000 when performing the simulation and computation. We refer this as the true pattern and it is displayed by the solid curve. For comparison, finite sample patterns are also provided with $T = 100, 250, 500, 1000$. 

12
The true pattern suggests that, when estimating jointly, one tends to get higher reserve levels than that estimated independently if $\rho \neq 0$. This is due to the behavior of the weighting function defined \((3.5)\). As illustrated in Figure \([4]\) for a non zero correlation $\rho$, the weighting function takes large values at both tails of $y_1$ and small values in the middle. As a consequence, the modified cdf has fatter tails than the marginal cdf. This is illustrated by the graph at the right side of Figure \([5]\) where the cdf of $N(0,1)$ (the shorter dashed line), and the modified cdf with $\rho = \pm 0.9$ (dashed and solid lines) are plotted.

Errors due to small sample size can be substantial. It is seen in Figure \([4]\) that, for sample sizes less than 500 (i.e. 2 years), the reserve level could be largely underestimated. This underestimation is diminished when sample size rises up to 1000 (i.e. 4 years).

ii) Constrained method

Let us now consider Method 3. Due to the two-step approach, the reserves computed with $y_{j,t}, j = 1, 2$ are no longer deduced by linear transformations of the reserves computed with $u_{j,t}, j = 1, 2$.

To highlight this effect, we consider returns $(y_{1,t}, y_{2,t})' \sim N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2 & \rho \sigma \\ \rho \sigma & 1 \end{pmatrix}\right]$, with a distribution depending on the correlation $\rho$ and the ratio of variances $\sigma^2$.

Figure \([5]\) shows a symmetric situation (i.e. $\sigma = 1$), where the objective function values are plotted against the proportion $\hat{z}_{1,t}^*(\alpha^*), \alpha/\hat{z}_{2,t}^*(\alpha^*)$, for $\rho = 0, 0.5, 0.9$. The reserve level allocated to asset 1 is roughly one half of the aggregate reserve level for modest correlation (say, $\rho < 0.6$). However, if correlation is large, it is optimal to assign more reserve to one asset than the other. To illustrate this formally, let us consider the limiting case where $\rho = 1$ (i.e. $y_{1,t} = y_{2,t}$ for all $t = 1, \ldots, T$). The FOC \((3.8)\) is satisfied if $z_{1,t}^* = z_{2,t}^* = z_t/2$. Evaluating at the same value of $z_{j,t}^*, j = 1, 2$, the second order condition (SOC) is given by:

$$
\frac{\partial^2 \Psi}{\partial z_1^2}(t, z_{1,t}^*, z_t - z_{1,t}^*) = -2E_t \left\{ \mathbb{I}_{y_{1,t+1} > z_{1,t}^*} \mathbb{I}_{y_{2,t+1} > z_t - z_{1,t}^*} \alpha_{11} + \mathbb{I}_{y_{1,t+1} < z_{1,t}^*} \mathbb{I}_{y_{2,t+1} < z_t - z_{1,t}^*} \alpha_{22} \right\} = -2 \left( \alpha_{11} Pr[y_{1,t} > \frac{z_t}{2}] + \alpha_{22} Pr[y_{1,t} < \frac{z_t}{2}] \right) < 0. \tag{3.9}$$

The FOC and SOC imply that a local maximum of the criterion value is achieved when $z_{1,t}^*/z_t = 0.5$.

Thus, the minimal value of the criterion function, if exist, must be a different solution $(\hat{z}_{j,t}^*, j = 1, 2)$ to the FOC \((3.8)\), where $\hat{z}_{1,t}^* \neq \hat{z}_{2,t}^* = z_t - \hat{z}_{1,t}^*$. If $\hat{z}_{1,t}^* < \hat{z}_{2,t}^*$, this implies:

$$
\alpha_{11} Pr[y_{1,t} > \frac{z_{2,t}^*}{2}] - \alpha_{12} Pr[z_{1,t}^* 1, t \leq y_{1,t} \leq \frac{z_{2,t}^*}{2}] + \alpha_{22} Pr[y_{1,t} < \frac{z_{1,t}^*}{2}] = 0. \tag{3.10}
$$

Given condition \((3.10)\), the SOC evaluated at $(\hat{z}_{j,t}^*, j = 1, 2)$ is given by:

$$
\frac{\partial^2 \Psi}{\partial z_1^2}(t, \hat{z}_{1,t}^*, z_t - \hat{z}_{1,t}^*) = (z_t - 2\hat{z}_{1,t}^* \alpha_{22} f(\hat{z}_{1,t}^*) - \alpha_{11} f(z_t - \hat{z}_{1,t}^*)) \tag{3.11}
$$
where $f$ denotes the pdf of $y_1$ (resp. $y_2$). In general, a positive SOC requires the following relationship of $(\alpha_{11}, \alpha_{22}, f, \text{and } z_t)$:

$$\alpha_{22} f(\tilde{z}_{1,t}^*) - \alpha_{11} f(z_t - \tilde{z}_{1,t}^*) > 0.$$  

Since $z_{1,t}$ and $z_{2,t}$ are interchangeable, the minimal point is not unique.

We report, in Figure 7, the relative optimal allocation $\hat{\tilde{z}}_{1,t}/\hat{\tilde{z}}_t$ as function of $\rho$ and $\sigma^2$. The data are generated with $\rho$ ranging from -0.9 to 0.9 and $\sigma^2 = \{1.5, 2, 2.5, 3\}$. When the value of correlation is small (say less than 0.5), we observe larger reserves are allocated to riskier business line. However, the reverse is true for strongly and positively correlated business lines. In both cases, the relative reserve allocation to riskier business line increases with the risk level.

4 Conclusion

The validation of reserve levels proposed by a bank often requires investigation of models’ forecasting accuracies of the VaR. A simple criterion consists in comparing the observed number of exceedances of the estimated VaR with the desired level. However, this criterion is not sufficient since the VaRs estimated by two completely different models may be exceeded by similar number of times in a finite sample. In this paper, we propose an alternative criterion which takes the level of loss beyond VaR into account. In the univariate case, this criterion coincides with the objective function of the standard quantile regression. We show that both the optimal criterion values and the values of lack of optimality can vary with the asset return volatility. A natural extension is conducted from the single line analysis to the risks with multiple lines. We explain how to allocate reserves to individual lines of risk with or without a constraint on global reserve.

References


**Appendices**

**Appendix A  The component for lack of optimality**

i) We have:

\[
\Psi(t, z_t) - \Psi(t, \hat{z}_t) = \alpha(\hat{z}_t - z_t) + E_t \left[ (z_t - y_{t+1}) 1_{z_t > y_{t+1}} \right] - E_t \left[ (\hat{z}_t - y_{t+1}) 1_{\hat{z}_t > y_{t+1}} \right].
\]

Let us consider the case \( \hat{z}_t > z_t \). We get:

\[
\Psi(t, z_t) - \Psi(t, \hat{z}_t) = \alpha(\hat{z}_t - z_t) + E_t \left[ (z_t - y_{t+1}) 1_{z_t > y_{t+1}} - (\hat{z}_t - y_{t+1}) 1_{\hat{z}_t > y_{t+1}} \right] - E_t \left[ (\hat{z}_t - y_{t+1}) 1_{\hat{z}_t > y_{t+1}} \right]
\]

\[
= \alpha(\hat{z}_t - z_t) + (z_t - \hat{z}_t) P_t \left[ z_t > y_{t+1} \right] - E_t \left[ (\hat{z}_t - y_{t+1}) 1_{\hat{z}_t > y_{t+1}} \right]
\]

\[
= (\hat{z}_t - z_t) E \left[ 1_{z_t < y_{t+1} < \hat{z}_t} \right] - E_t \left[ (\hat{z}_t - y_{t+1}) 1_{\hat{z}_t < y_{t+1}} \right]
\]

\[
= E_t \left[ (y_{t+1} - z_t) 1_{z_t < y_{t+1} < \hat{z}_t} \right].
\]

A similar computation can be done when \( z_t > \hat{z}_t \).

ii) We have:

\[
E_t \left[ (y_{t+1} - z_t) 1_{z_t < y_{t+1} < \hat{z}_t} \right] = \int_{z_t}^{\hat{z}_t} (y - z_t) d F_t(y)
\]

\[
= (y - z_t) F_t(y) \bigg|_{z_t}^{\hat{z}_t} - \int_{z_t}^{\hat{z}_t} F_t(y) dy
\]

\[
= (\hat{z}_t - z_t) F_t(\hat{z}_t) - \int_{z_t}^{\hat{z}_t} F_t(y) dy
\]

\[
= \int_{z_t}^{\hat{z}_t} \left[ F_t(\hat{z}_t) - F_t(y) \right] dy.
\]

iii) Finally, if \( z_t \simeq \hat{z}_t \), we get:

\[
\int_{z_t}^{\hat{z}_t} \left[ F_t(\hat{z}_t) - F_t(y) \right] dy \simeq \int_{z_t}^{\hat{z}_t} f_t(\hat{z}_t)(\hat{z}_t - y) dy \simeq \frac{1}{2} f_t(\hat{z}_t)(\hat{z}_t - z_t)^2.
\]
Appendix B  Weight correction

Let us assume that $y_{t+1} = m_t + \sigma_t u_{t+1}$, where the error term has a path independent distribution with pdf $g$ and cdf $G$. The conditional pdf and cdf of $y_{t+1}$ are respectively given by:

$$f_t(y) = \frac{1}{\sigma_t} g \left( \frac{y - m_t}{\sigma_t} \right), F_t(y) = G \left( \frac{y - m_t}{\sigma_t} \right).$$

We deduce:

$$f_t \left[ F_t^{-1}(\alpha) \right] = f_t \left[ m_t + \sigma_t G^{-1}(\alpha) \right] = \frac{1}{\sigma_t} g \left[ G^{-1}(\alpha) \right].$$
Table 1: Summary statistics

<table>
<thead>
<tr>
<th>Series</th>
<th>Mean</th>
<th>Variance</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi(t, \tilde{z}_t)$</td>
<td>0.0926</td>
<td>0.0005</td>
<td>0.0762</td>
<td>2.4029</td>
</tr>
<tr>
<td>$c(t, z_t)$</td>
<td>0.0178</td>
<td>0.0003</td>
<td>1.0883</td>
<td>3.5836</td>
</tr>
<tr>
<td>$\psi(t, z)$</td>
<td>0.1104</td>
<td>0.0003</td>
<td>2.6492</td>
<td>10.8276</td>
</tr>
</tbody>
</table>
Table 2: Autocorrelations

<table>
<thead>
<tr>
<th>Series</th>
<th>( \psi(t, \hat{z}_t) )</th>
<th>( c(t, z_t) )</th>
<th>( \psi(t, z) )</th>
<th>Series</th>
<th>( \psi(t, \hat{z}_t)^2 )</th>
<th>( c(t, z_t)^2 )</th>
<th>( \psi(t, z)^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho(0) )</td>
<td>1.0000</td>
<td>-0.7024</td>
<td>0.6730</td>
<td>( \psi(t, \hat{z}_t)^2 )</td>
<td>1.0000</td>
<td>-0.5363</td>
<td>0.7420</td>
</tr>
<tr>
<td>( \rho(1) )</td>
<td>-0.6725</td>
<td>0.0964</td>
<td>-0.0083</td>
<td>( c(t, z_t)^2 )</td>
<td>-0.5193</td>
<td>0.8856</td>
<td>0.0170</td>
</tr>
<tr>
<td>( \rho(2) )</td>
<td>0.9562</td>
<td>-0.6681</td>
<td>0.6465</td>
<td>( \psi(t, \hat{z}_t)^2 )</td>
<td>0.5951</td>
<td>-0.5136</td>
<td>0.7008</td>
</tr>
<tr>
<td>( \rho(3) )</td>
<td>-0.6489</td>
<td>0.8108</td>
<td>-0.0778</td>
<td>( c(t, z_t)^2 )</td>
<td>-0.5109</td>
<td>0.7891</td>
<td>-0.0496</td>
</tr>
<tr>
<td>( \rho(4) )</td>
<td>0.5644</td>
<td>-0.0558</td>
<td>0.7777</td>
<td>( \psi(t, \hat{z}_t)^2 )</td>
<td>0.6417</td>
<td>-0.0248</td>
<td>0.7306</td>
</tr>
<tr>
<td>( \rho(5) )</td>
<td>0.8657</td>
<td>-0.6006</td>
<td>0.5877</td>
<td>( \psi(t, \hat{z}_t)^2 )</td>
<td>0.8630</td>
<td>-0.4713</td>
<td>0.6161</td>
</tr>
<tr>
<td>( \rho(6) )</td>
<td>0.5719</td>
<td>-0.0794</td>
<td>0.7054</td>
<td>( \psi(t, \hat{z}_t)^2 )</td>
<td>0.8289</td>
<td>-0.4304</td>
<td>0.6159</td>
</tr>
<tr>
<td>( \rho(7) )</td>
<td>0.5621</td>
<td>-0.0491</td>
<td>0.7184</td>
<td>( \psi(t, \hat{z}_t)^2 )</td>
<td>0.5996</td>
<td>-0.0214</td>
<td>0.6906</td>
</tr>
<tr>
<td>( \rho(8) )</td>
<td>0.7888</td>
<td>-0.5131</td>
<td>0.5806</td>
<td>( \psi(t, \hat{z}_t)^2 )</td>
<td>0.8048</td>
<td>-0.3997</td>
<td>0.6220</td>
</tr>
<tr>
<td>( \rho(9) )</td>
<td>-0.5364</td>
<td>0.6795</td>
<td>-0.0606</td>
<td>( c(t, z_t)^2 )</td>
<td>-0.4301</td>
<td>0.6455</td>
<td>-0.0219</td>
</tr>
<tr>
<td>( \rho(10) )</td>
<td>0.5571</td>
<td>-0.0606</td>
<td>0.7502</td>
<td>( \psi(t, \hat{z}_t)^2 )</td>
<td>0.6037</td>
<td>0.0171</td>
<td>0.7464</td>
</tr>
<tr>
<td>( \rho(11) )</td>
<td>0.6082</td>
<td>-0.0421</td>
<td>0.4263</td>
<td>( \psi(t, \hat{z}_t)^2 )</td>
<td>0.5942</td>
<td>-0.3417</td>
<td>0.4364</td>
</tr>
<tr>
<td>( \rho(12) )</td>
<td>-0.4448</td>
<td>0.4699</td>
<td>-1.092</td>
<td>( c(t, z_t)^2 )</td>
<td>-0.3770</td>
<td>0.5445</td>
<td>-0.0973</td>
</tr>
<tr>
<td>( \rho(13) )</td>
<td>0.3796</td>
<td>-0.0591</td>
<td>0.4436</td>
<td>( \psi(t, \hat{z}_t)^2 )</td>
<td>0.4076</td>
<td>-0.0459</td>
<td>0.4185</td>
</tr>
<tr>
<td>( \rho(14) )</td>
<td>0.4237</td>
<td>-0.2383</td>
<td>0.3302</td>
<td>( \psi(t, \hat{z}_t)^2 )</td>
<td>0.4124</td>
<td>-0.2166</td>
<td>0.3183</td>
</tr>
<tr>
<td>( \rho(15) )</td>
<td>-0.3655</td>
<td>-0.2752</td>
<td>-0.2140</td>
<td>( c(t, z_t)^2 )</td>
<td>-0.3058</td>
<td>0.3453</td>
<td>-0.1515</td>
</tr>
<tr>
<td>( \rho(16) )</td>
<td>0.2239</td>
<td>-0.0322</td>
<td>0.2545</td>
<td>( \psi(t, \hat{z}_t)^2 )</td>
<td>0.2471</td>
<td>-0.0412</td>
<td>0.2278</td>
</tr>
<tr>
<td>( \rho(17) )</td>
<td>0.0286</td>
<td>-0.0163</td>
<td>0.0215</td>
<td>( \psi(t, \hat{z}_t)^2 )</td>
<td>0.0064</td>
<td>0.0601</td>
<td>-0.0077</td>
</tr>
<tr>
<td>( \rho(18) )</td>
<td>-0.0884</td>
<td>0.0626</td>
<td>-0.0580</td>
<td>( c(t, z_t)^2 )</td>
<td>-0.1541</td>
<td>0.0636</td>
<td>-0.0841</td>
</tr>
<tr>
<td>( \rho(19) )</td>
<td>-0.1145</td>
<td>0.0907</td>
<td>-0.0685</td>
<td>( \psi(t, \hat{z}_t)^2 )</td>
<td>-0.1011</td>
<td>0.1283</td>
<td>0.0740</td>
</tr>
</tbody>
</table>

Table 3: Test results

<table>
<thead>
<tr>
<th>Cases</th>
<th>( SSR_T )</th>
<th>( c )</th>
<th>Results</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>13.0939</td>
<td>2.7056</td>
<td>Y</td>
</tr>
<tr>
<td>II</td>
<td>14.7373</td>
<td>4.9471</td>
<td>Y</td>
</tr>
<tr>
<td>III</td>
<td>14.4964</td>
<td>4.7363</td>
<td>Y</td>
</tr>
<tr>
<td>IV</td>
<td>15.1493</td>
<td>5.9920</td>
<td>Y</td>
</tr>
</tbody>
</table>

The test statistics \( SSR_T \) are provided in column 2. Column 3 lists the critical values. When \( K \leq 2 \), we obtain the values of \( c \) by using the closed form expressions of the distribution of \( SSR_T \) and a grid search numerical algorithm. Otherwise, the weights \( \pi_k \) are obtained through a simulation procedure. The letter “Y” means that the test concludes that the estimated optimal procedure \( \hat{z}_t \) is better than the suboptimal procedure \( \hat{z}_t \).
Figure 1: Joint Simulated Path of \((r_t, \sigma^2_t)\)

Figure 2: Conditional and Unconditional Quantiles
Figure 3: Joint Evolution of $\Psi(t, \hat{z}_t)$, $\Psi(t, z)$ and $c(t, z_t)$

Figure 4: Disaggregate reserve level, function of $\rho$

21
Figure 5: Value of weights as function of \((\rho, y_1)\) and modified cdf when \(\rho = \pm 0.9\)

Figure 6: Disaggregate reserve level as function of \(\rho\) with aggregate reserve constraint
Figure 7: Relative allocation as function of $\rho$ and $\sigma^2$